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A Note on Generalized Möbius-Listing Bodies

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ABSTRACT

Generalized Möbius-Listing surfaces and bodies generalize Möbius bands, and this research was motivated originally by solutions of boundary value problems. Analogous to cutting of the original Möbius band, for this class of surfaces and bodies, results have been obtained when cutting such bodies or surfaces. In general, cutting leads to interlinked and intertwined different surfaces or bodies, resulting in very complex systems. However, under certain conditions, the result of cutting can be a single surface or body, which reduces complexity considerably. These conditions are based on congruence and rotational symmetry of the resulting cross sections after cutting, and on the knife cutting the origin.

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1. INTRODUCTION

The notion of *Generalized Möbius-Listing Surfaces and Bodies* GML was pioneered by the second author, in collaboration with Paolo Emilio Ricci and joined by the first author from 2010 [1,2,3,4]. The motivation arose from a clever remark by the famous Russian mathematician Olga A. Oleinik, who observed that the solutions of BVP for partial differential equations strongly depend on the topological properties of the domain in which the problem is considered. A second motivation was to decompose complex movements into simple movements, following the original idea of Gaspar Monge in the late 18th, early 19th century.

GML's generalize the Möbius band or ribbon in a geometrical way: both the basic line and the cross section can be chosen from a wide range of shapes, compared, in the case of the classical Möbius band, to the circle as central basic line and a piece of line perpendicular to the basic line; moreover, the basic line cuts the piece of line exactly in the middle (Fig. 1a).

Fig. 1 shows how a ribbon can be connected, with an odd or even number of twists (the upper index). In case the number of twists is odd, the surface is called non-orientable, and the surface is one-sided. In other words, a traveler moving on the Möbius band will return at his or her original position and will have visited all parts of the ribbon ("Möbius condition"). In contrast, when the number of twists is even, then the surfaces will be like a cylinder, with an inner and an outer surface ("Cylinder condition"). A traveler starting on one side will return on the same side, and not see the other side.

Instead of starting from a ribbon, GML start from prisms (or cylinders), of which the two ends are connected. The notation is GML_m^n where the lower index refers to the cross sectional symmetry (e.g. $m = 6$ for a hexagon or a star with six points, Fig. 2c and Fig. 2d), and if before joining the prism is twisted, then the number of twists (relative to the symmetry m) is given by the upper index (if $n = 0$ there are no twists). In the case of ribbons, the lower index is $m = 2$ whereas the value of the upper index n describes the number of rotations by 180° before joining the ends (Fig. 1).

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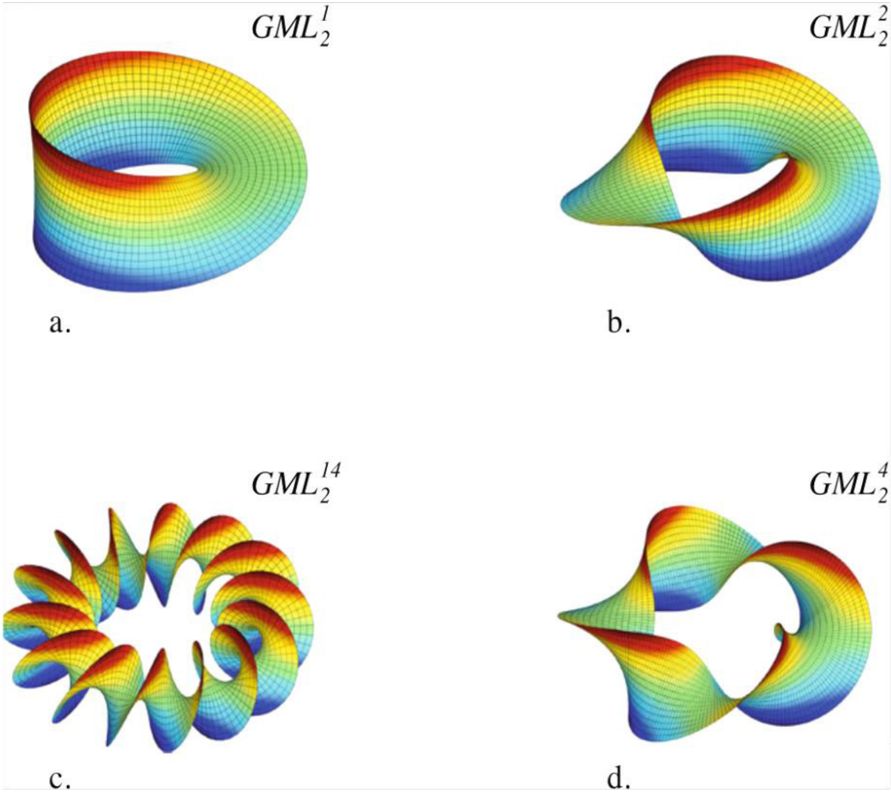


Figure 1. Ribbons with twists before joining. Case a (top left) is the classic Möbius strip.

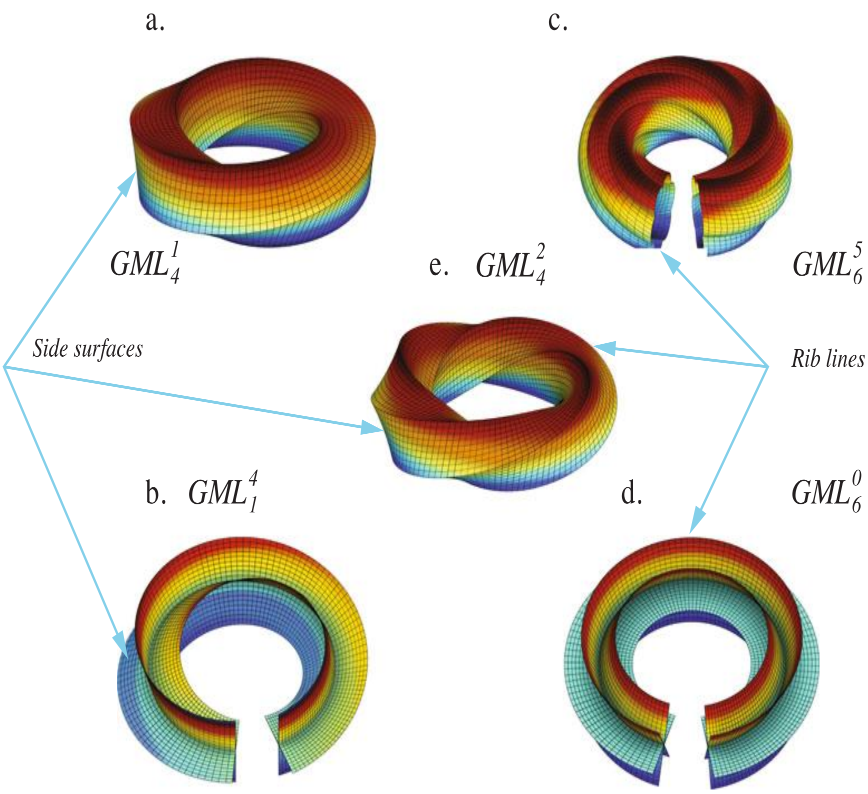


Figure 2. (a-e) Identification of vertices, with twists leading to torus GML_m^n.

Conceptually GML_m^n are straightforward, creating both possibilities and challenges. In the past decades, the focus has been on:

- the analytic definition of GML_m^n;
- the process of cutting and its general solution;
- the conditions under which a single body or surface results after cutting;
- the wide applicability.

2. ANALYTIC DEFINITION

Generalized Möbius-Listing Bodies GML_m^n are defined analytically by:

$$\begin{cases} X(\tau,\psi,\theta,t) = T_1(t) + \left[R(\theta,t) + p(\tau,\psi,\theta,t)\cos(\psi + \frac{n\theta}{m})\right]\cos(\theta + M(t)) \\ Y(\tau,\psi,\theta,t) = T_2(t) + \left[R(\theta,t) + p(\tau,\psi,\theta,t)\cos(\psi + \frac{n\theta}{m})\right]\sin(\theta + M(t)) \\ Z(\tau,\psi,\theta,t) = T_3(t) + Q(\theta,t) + p(\tau,\psi,\theta,t)\sin\left(\psi + \frac{n\theta}{m}\right) \end{cases} \tag{1}$$

The functions $T_{1,2,3}(t), R(\theta,t), p(\tau,\psi,\theta,t), M(t)$ and $Q(\theta,t)$, as well as parameter $\mu = \frac{n}{m}$, define simple movements. The most important are:

- Shape of the cross section $p(\tau,\psi,\theta,t)$.
- Shape of the basic line: the trajectory that the central point of the cross section follows before closing. $R(\theta,t)$.
- The number of twists $\mu = \frac{n}{m}$.
- Translation of the body in 3D space $T_{1,2,3}(t)$.
- In the case of closed GML bodies, the function Q(θ,t) is 2π-periodic or equal to 0.

Some remarks:

This baseline and the cross section can be a circle, Gielis curve, polygon, ..., even a self-intersecting curve (Fig. 3a).

- If the cross section is a curve, then the resulting shapes will be a GML surface (hollow), but if also the area inside the curve is included, the resulting shapes will be a GML body (solid). Obviously, one can also think of shells, i.e. GML bodies with an outer thickness and the innermost part hollow, like a bamboo culm.
- In [5], the problem of how many different hollow bodies results when the cross sections are Grandi curves, also known as rose curves, defined by the control parameters $\{p,q\}$. This question concerns the number of fluids that can be contained in a GML_m^n body.
- The pure GML_m^n bodies and surfaces are always closed (Fig. 3a, Fig. 3e), but they are a subset of Generalized Twisting and Rotating bodies GTR_m^n (Fig. 3b, Fig. 3c, Fig. 3d, Fig. 3f). The basic line can be for example a helix (Fig. 3b, Fig. 3c) or a spiral (Fig. 3d, Fig. 3f), with non-closed shapes.
- With this analytic representation, complex movements can be studied and decomposed into simple ones, allowing for the study of this class of surfaces and bodies from a kinematic point of view. However, the same parameterizations also lend themselves to dynamic studies, with changing cross sections in time (hence the parameter t).
- Other analytic representations exist [6], but this one is the simplest.

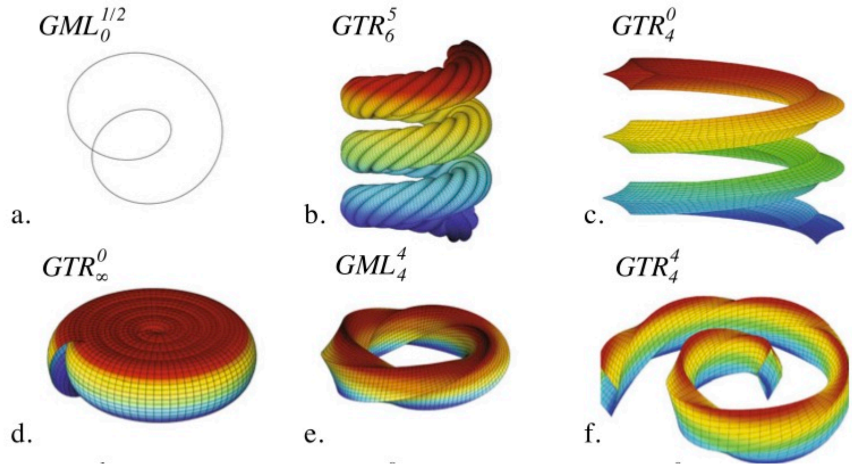


Figure 3. *GML* and *GTR* surfaces.

3. THE CUTTING PROBLEM

The original motivation to study GML_m^n surfaces and bodies is that the solution of boundary value problems for partial differential equations is easier to obtain with direct knowledge of the domain. Further, with the extension of surfaces to bodies, also of the internal geometry and connected domains inside GML_m^n bodies, to understand the precise relation between the asymptotic behavior of solutions and the geometrical structure. A number of studies have focused on elasticity of Möbius bands only, but the general problem is open.

To address this problem, the idea was to study cutting the GML_m^n structure with a knife. One can imagine a knife moving along the GML_m^n until the knife returns to its original position. In the case of a classic Möbius band, the results depend on where the cut is made. If the Möbius band is cut along the basic line, a single surface results. If it is cut along any other straight line, the result is a link of two different surfaces. The second author generalized this result for any n and m, and for any number of knives [3].

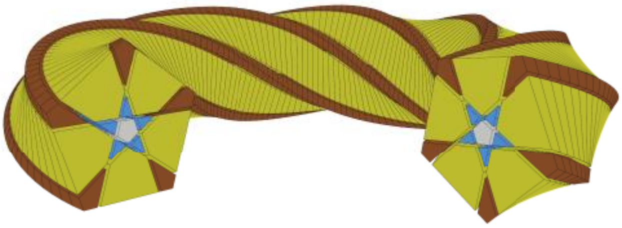
When dealing with GML surfaces and bodies, more objects can result after cutting. Depending on the where precisely the knife cuts (through the center of the polygon or not; from vertex to vertex; from side to side, or from vertex to side, and on the m and the number of twists of the GML_m^n) many different outcomes result [1,2,3,4,6,7,8]. The knife keeps this fixed position (e.g. vertex 1 to vertex 3) from start to finish. Without loss of generality, we choose regular polygons as cross sections. For example, in the case of a GML with a pentagonal cross section (Fig. 4), after cutting from side to side the structure falls apart into four different pieces (indicated by different colors), which remain completely intertwined after separation. An example is shown in Fig. 4 top right.

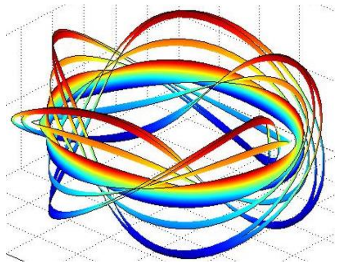
The general solution was found when the whole problem could be reduced to pure planar geometry. Instead of using a fixed knife to cut the fixed GML surface or body, the knife can be fixed and the GML moves. It is described in [8], but the inspiration was machinery for cutting bamboo poles [9].

Remarks:

- The choice for cutting is to be understood in a broad sense. It is about *separation* of different domains within larger domains when these are subject to stresses.
- When a prism is subject to torsion, the cross sections will not remain planar (they will remain planar for a rod with circular cross section). Part of the section will be forced upward, other parts downwards. But the "nodal line", which remains in plane, separates the two. Fig. 5 shows an example by Saint-Venant who discovered this (Fig. 5a, Fig. 5b) and D'Arcy Thompson's plaster cast of horns or rams (Fig. 5c). Similar for vibrations. In Chladni plates, sand gathers where stress and deformation are zero, at the nodal lines (Fig. 5d).
- Whether we speak of drawing lines to connect points and separate subdomains, using knives to cut bodies or m-polygons, considering paths at 1000 m of height as the only way to avoid falling into volcano craters, they are all the same.

- The problem and its solution also generalized many other classic problems in geometry studied by Euler, Catalan, Lamé, Cayley and many others [9].





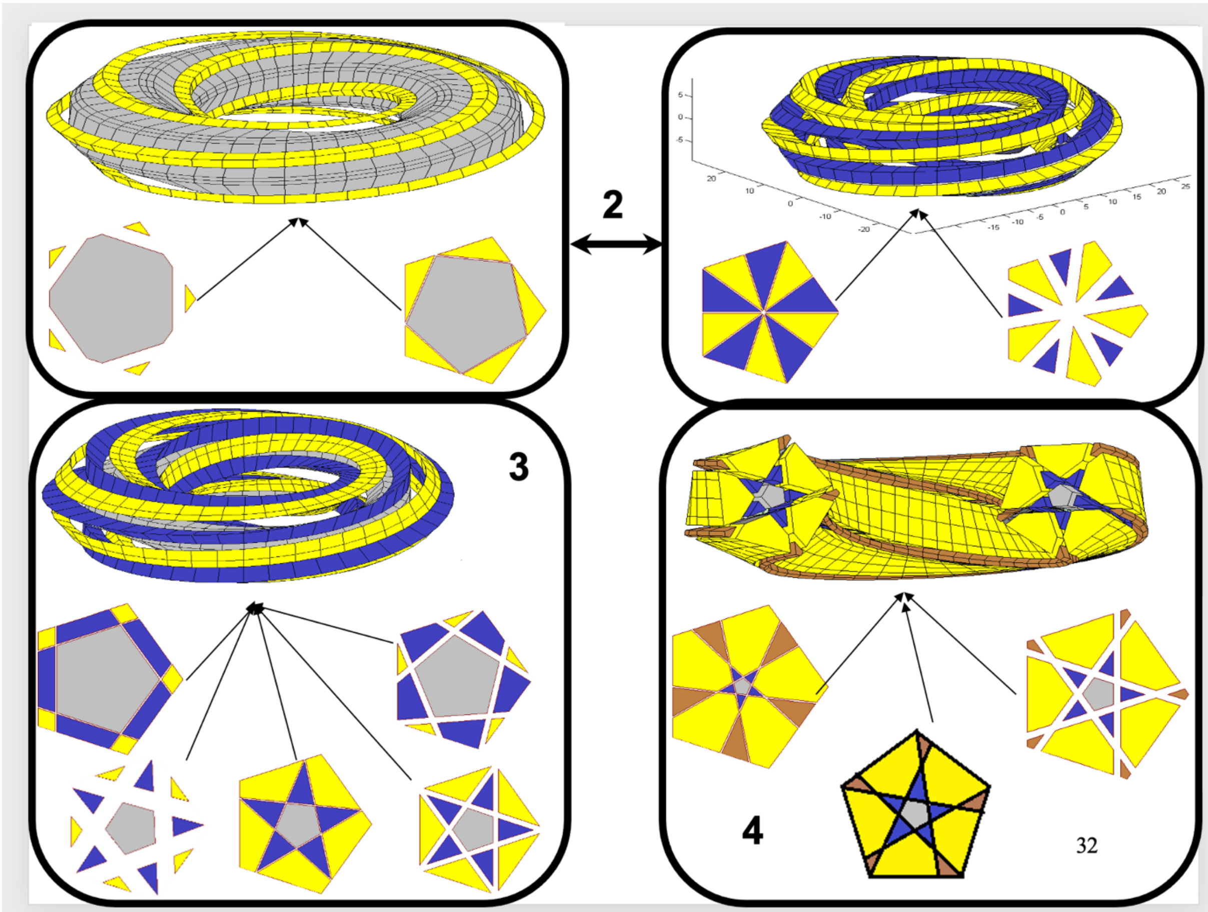
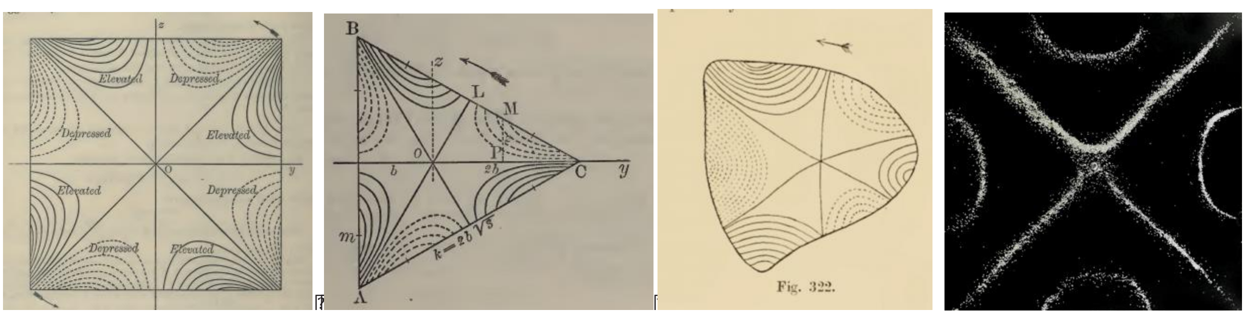


Figure 4. A *GML* body with pentagonal cross section and resulting structures after cutting.



(a) (b) (c) (d)

Figure 5. Torsion of prisms with square (a) and triangular (b) cross sections. Dotted lines are depressed regions and full lines are elevated parts of the cross sections. (c) Anticlastic surfaces in the horns of rams. (d) Example of Chladni plate.

4. HOW TO OBTAIN A SINGLE SURFACE OR BODY AFTER CUTTING

In studying the cutting problems, very complex structures can result, but under certain conditions, a single body will result. This is similar to the classic Möbius band: when cut along the central line, the result is a single one-sided ribbon. In GML it was found that the same results can be obtained when cutting the polygon through the center for *m* even.

But a twist is necessary. Fig. 6 shows all possible results after cutting an untwisted and a twisted square torus. Here $SS_{1,2}$ and $SS_{1,3}$ indicate that the knife cuts from side 1 to sides 2 and 3 respectively. Likewise, VS stands for vertex to side cuts and VV for vertex to vertex cuts. In cases B and D, the cut is through the center of a GML body with square cross sections; the result is always two bodies. However, if given a twist of 180°, or multiples of 180°, before joining the ends, then a single body results after cutting (indicated by Link-1 in Link group of objects in Fig. 6b). It happens when the cut is through the center (cases B.II and D in Fig. 6b).

Hence we find that conditions must exist under which the whole structure does not divide into different pieces, but ultimately remains one connected structure. And this structure displays the Möbius phenomenon (as opposed to the Cylinder phenomenon). Given that the cross section can change over time (or along the length of the basic line), the cross sections can become very complicated. But still, the overall result will be a single body under the appropriate conditions.

The next challenge was then to find these conditions. With the process of cutting described above, with a knife cutting from side to side, vertex to vertex or side to vertex, it turned out that a single body could only be obtained for polygons with even number of vertices and edges, and only when the cut was through the center (cases B.II and D in Fig. 6b). It did not work if the cut was not through the center (Cases A, B.I and C in Fig. 6b).

The difference between B.II and D on the one hand and case C on the other, is that the resulting shapes of the former (the grey and yellow) are congruent, and they can be rotated around the center, so that the yellow completely overlaps the grey after rotation around the center. Hence, congruence is key, but this is not possible at all, when the number of sides and edges is odd (triangles, pentagons or heptagons). Moreover, this rotational symmetry condition is met in Fig. 7a and Fig. 7b, but not in Fig. 7c. So, if the shape has cross section **c.**, then the single body phenomenon is not possible either. Thus, it is also not general for *m* even.

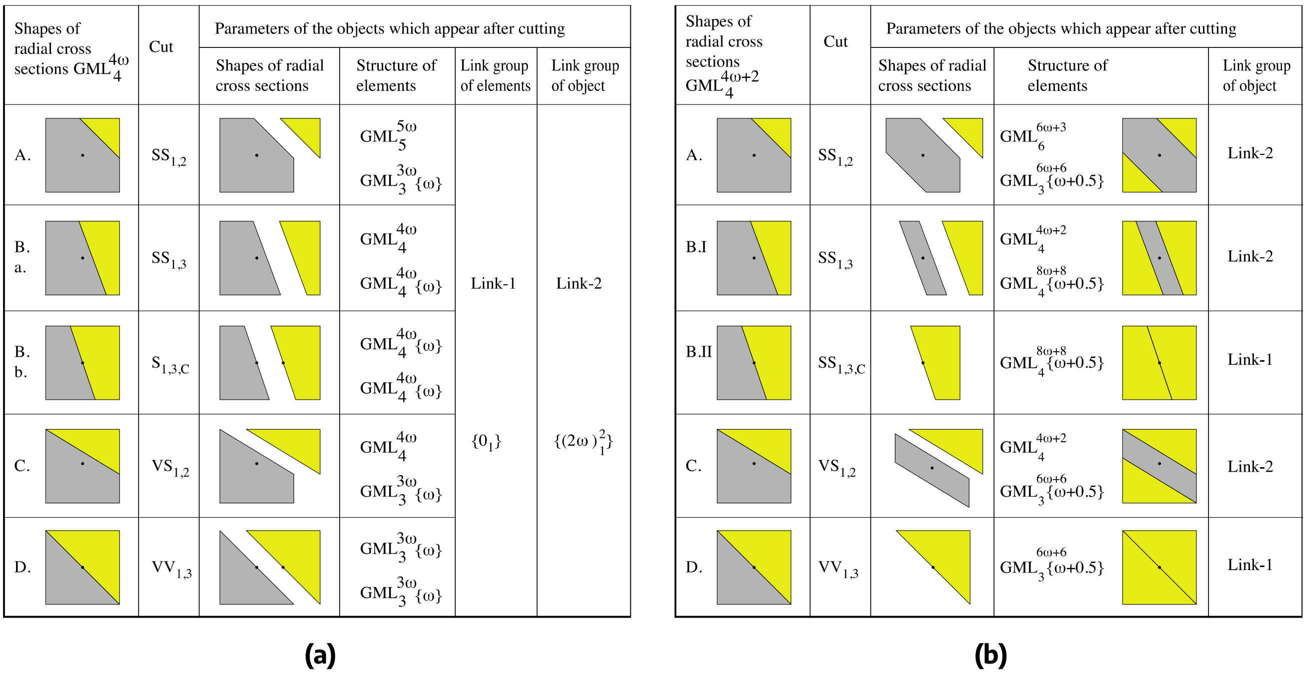


Figure 6. Cutting of GML_4^n bodies, without (a) or with (b) twist.

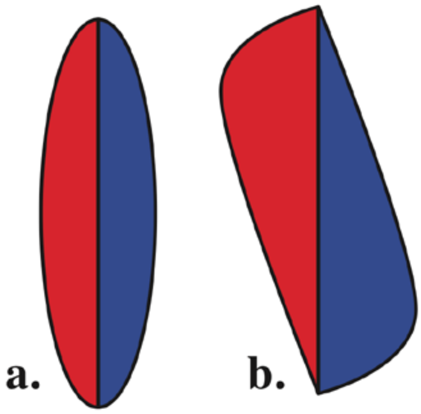


Figure 7. Two angular bodies cut through the center.

This problem was solved using radial knives, which cut only from center to boundary, not the whole way through like the previous knives. But if one completes the cutting along the whole structure until the radial knife returns to its original position, then the single-sided Link-1 Möbius phenomenon is possible also for polygons with odd number of vertices and sides. This time the idea of using a radial knife was inspired by sawing wood. Details are found in [8], but Fig. 8 shows how congruence by rotation is achieved by radial knives in (b) and (d), but not with the cross-cutting knives in (a) and (c). Here the dark and light blue sectors are mirrored, not congruent by simple rotation.

5. A SINGLE CROSS SECTION WITH THE RIGHT SYMMETRY SUFFICES

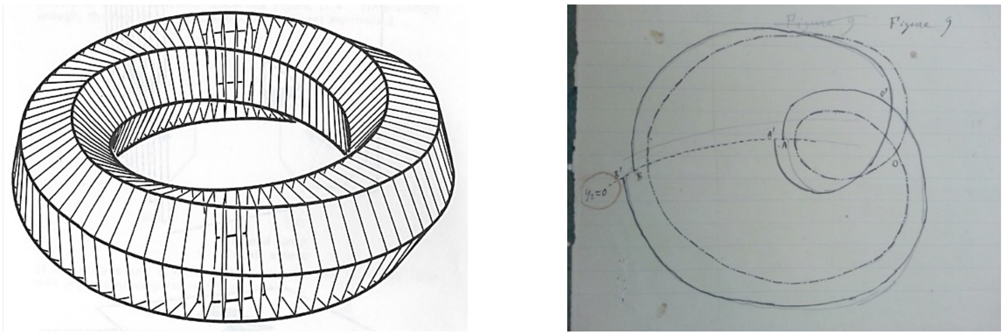
The cross section can change over time ($R(\theta, t)$ in Eq. (1)). It may start as a square, but morph into a pentagon, before returning back to a square, whereby the end cross section is the starting cross section. What is in between the two cross sections can actually be almost anything. A natural question then is how much of the cross sections needs to fulfil the conditions? Along the whole structure? Along three quarters? Half? Or just a short piece?

The simple answer is that a single cross section is sufficient. The cross section of the rest of the GML body can be point clouds or the wildest asymmetrical shapes you can imagine, but as long as the cutting process starts with one cross section fulfilling the conditions (which is obviously then also the end cross section), that is fine [10].

Fig. 9 illustrates this, as an old carousel slide tray, but with a twist. If we start with one cross section having the right conditions for cutting (which will also be the end condition), then all of the other cross sections can be anything. If this is a simple hexagon with uniform color, the rest can be anything inside the hexagonal boundary (in the case the size of the cross section remains the same along the structure, we speak of area conservation). But these cross sections can be anything. It does not matter. For example, if the starting section is a hexagon with a nice symmetrical figure, then this figure can be torn into pieces, or into point clouds which may reach to the end of the universe, but when the knife reaches its final destination (i.e. the symmetrical shape at the starting position), the whole structure turns out to be one simple connected body.

(a) (b) (c) (d)

Figure 8. Vertex-to-side and side-to-side cuts through the origin in a triangle, with chordal (a,c) or radial (b,d) knives.



(a) (b)

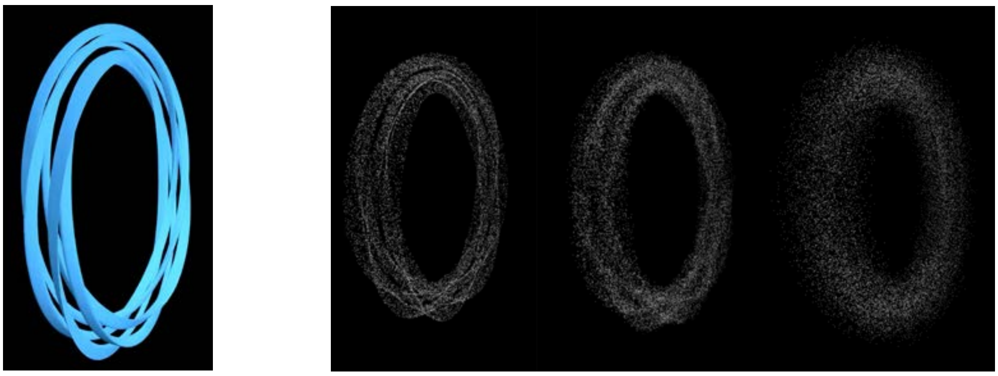
Figure 9. Left: fiber bundle. Right: Poincaré's drawing.

Fig. 9a is from a book by J. Weeks, *The Shape of Space*, illustrating fiber bundles in differential geometry. This gives an idea of how far our GML results can extend. One aspect of dynamical systems that is particularly relevant is Poincaré's Recurrence Theorem. In many physical and mathematical cases conservation of area (or volume) is crucial. In Fig. 9a, all sections have the same area, but what the hexagons contain can be very different. One example with area (or volume) conservation is Poincaré's Recurrence Theorem, which states that under certain conditions any dynamical system will return arbitrarily close to its initial state after a certain time (Fig. 9b). More precisely, Poincaré proved that in certain systems almost all trajectories return arbitrarily close to their initial position infinitely often. What exactly the intermediate trajectories are is less important.

Arnold's Cat is the most famous illustration of the phenomenon. Starting with the image of a cat, and performing an area preserving transformation eventually, after a specific number of discrete steps, the image of the cat completely reappears, and will continue to do so, infinitely. The transformation tears up the image, but maps it back to fit the original square in every step, hence it is area-preserving. This can be done with other shapes as well, e.g. plant leaves [11]. Obviously, in a GML body the cat or the leaves may be one cross section (beginning and end) and all other cross sections can be the intermediate discrete steps and are of the same area as the initial one, since the transformation is area-preserving. If the image has rotational symmetry, one can cut the GML body through the center and the result will be a single, one-sided body.

6. FUTURE PROSPECTS

There are a number of applications in science and technology. They can be models for a wide range of complex phenomena, but then, first and foremost, it is necessary to understand the geometry of GML_m^n bodies to be able to observe these phenomena. Fig. 10a shows a GML_5^n body in blue. The other panels show the same body represented by point clouds, with increasing noise, which can be representative of real physical phenomena, with measurement uncertainty.



(a) (b) (c) (d)

Figure 10. (a) GML_5^n body. (b-d) The same body represented by point clouds, with increasing noise.

It will depend on the accuracy of the measurement, whether such structures can be observed. With decreasing measurement precision and no knowledge about GML_m^n structures, one might easily mistake data points resulting from measurements as toroidal (Fig. 10d), or a having some structure (Fig. 10b, Fig. 10c). This "measurement" problem will be even worse if one has only datapoints from 2D-cross sections. The structure in Fig. 10b may be interpreted as resulting from phenomena with a non-linear resonances footprint, whereas the system is completely deterministic.

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