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A New Approach to Circular Inversion in l_1-Normed Spaces

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Abstract: While there are well-known synthetic methods in the literature for finding the image of a point under circular inversion in l_2-normed geometry (Euclidean geometry), there is no similar synthetic method in Minkowski geometry, also known as the geometry of finite-dimensional Banach spaces. In this study, we have succeeded in creating a synthetic construction of the circular inversion in l_1-normed spaces, which is one of the most fundamental examples of Minkowski geometry. Moreover, this synthetic construction has been given using the Euclidean circle, independently of the l_1-norm.

Keywords: finite-dimensional Banach spaces; Minkowski geometry; metric geometry; l_1-norm; Manhattan metric; taxicab metric

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1. Introduction

The main geometrical methods in the sciences go back to ancient times and are based on lines and circles. The ubiquitous contemporary use of Fourier methods in physics and engineering is mathematically equivalent to the methods used by Ptolemy [1].

Euclidean geometry has mainly evolved into Riemannian geometry, with local Euclidean geometry in tangent spaces [2]. Another extension is into Minkowski–Finsler geometry, also referred to as Riemannian geometry without the quadratic restriction [3]. Just as Euclidean geometry in two dimensions is based on using the circle as the basic figure for the isotropic determination of distances, Lamé curves (also referred to as supercircles and superellipses) form the basis of the simplest definitive Minkowski–Finsler geometries. Generalizing superellipses to any symmetry (Gielis transformations) extends the methods that the natural sciences have to describe various natural anisotropies for different symmetries s ($s = 0, 1, 2, 3, 4, 5$, etc., or any real value), as observed in starfish or diatoms [4]. By applying Gielis transformations to the most “natural” curves and surfaces of Euclidean geometry, e.g., circles under closed curves and logarithmic spirals under non-closed curves in two dimensions, one obtains many shapes observed in nature, biology, crystallography, physics, and other fields [4,5].

Over the past decade, these curves and transformations have been used successfully to model of a wide variety of natural shapes, from plant leaves and tree rings to starfish and avian eggs [6]. Further applications of Finsler geometry involving superellipses or Gielis curves in the natural sciences are found in forest ecology [7,8], seismic ray paths in anisotropic media [9], and the spreading of wildfires [10]. This has inspired various researchers to study inversions, using classical inversion over a circle (Figure 1) [11], or with Lamé–Gielis curves as inversion circles [12,13].

Inversions are area-preserving transformations, whereby the area of a rectangle formed by two distances from the center is equal to the area of the square on the radius of the circle (see [14] for more details). This is very similar to the parabola, a machine for transforming rectangles into squares with the same area. In Euclidean geometry, parabolas, ellipses, and hyperbolas are related directly to the application of areas in Euclid’s Book II. Since the basic shapes of our simplest Minkowski–Finsler geometries are a one-step generalization of the classic conic sections, it is of great interest to return to the methods used in ancient times. While, in Euclidean geometry, the image of a point under circular inversion can be found by a synthetic method, in another type of geometry that *stands next to and is a relative of Euclidean geometry (called Minkowki geometry)*, synthetic methods for circular inversion are not well known.

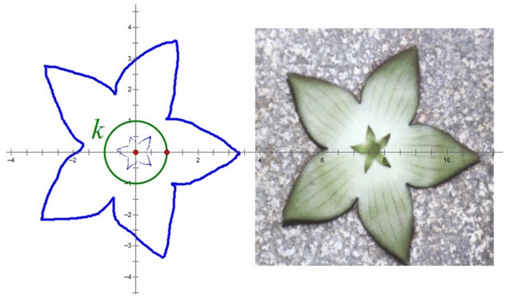

Figure 1. Huernia flowers and how the outer edges of the corolla (blue) are obtained by the inversion of the sepals (light blue) over the green circle k (see [11] for details).

The concept of Euclidean distance can be generalized to any linear space by the introduction of a norm. In this sense, the geometry that reveals, with the help of norms, infinite-dimensional Banach spaces is called Minkowski geometry (we refer the reader to [15–17] for a wider treatment of this). This is often confused with spacetime geometry, also called Minkowski geometry. Hermann Minkowski (1864–1909) was a pioneer in both geometries, and the general structure of Minkowski space was introduced by Minkowski while working on some problems in number theory [18]. However, considering that Riemann mentioned the l_4-norm in [19], it can be said that the first step towards Minkowski geometry was taken by Riemann. A norm on a vector space V is a function $\rho : V \to \mathbb{R}$ for which the following hold for any $\alpha, \beta \in V$, and $\lambda \in \mathbb{R}$:

i. $\rho(\alpha) \geq 0,\ \forall \alpha \in V$ (positive definite, $(\rho(\alpha) = 0\ iff\ \alpha = \overrightarrow{0}\,)$);
ii. $\rho(\lambda\alpha) = |\lambda|\rho(\alpha)$ (positive homogeneity);
iii. $\rho(\alpha + \beta) \leq \rho(\alpha) + \rho(\beta)$ (triangle inequality).

A normed space (V, ρ) is a set V with a norm ρ defined on V. The most important thing that explains the relationship between normed spaces and metric spaces is the following proposition:

Lemma 1. *A normed linear space (V, ρ) is a metric space with the distance*

$$d\rho(\alpha, \beta) := \rho(\beta - \alpha).$$

Proof. The metric properties (non-negative and symmetry) of $d\rho$ are quite obvious from the norm axioms. From the triangular inequality property of the norm,

$$\rho(\alpha - \gamma) = \rho(\alpha - \beta + \beta - \gamma) \leq \rho(\alpha - \beta) + \rho(\beta - \gamma)$$

for any α, β, or $\gamma \in V$. □

Also, the most well-known examples of normed spaces are the l_p-normed space and α-normed space, whose definitions are given below.

Let $\overrightarrow{X} = (x_1, \ldots, x_n)$ be a vector in the n-dimensional real vector space $\mathbb{R}^n$. The l_p-norm is defined as

$$\left\|\overrightarrow{X}\right\|_p := \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} & , \ p \in [1,\infty) \\ \max_{1\le i\le n} |x_i| & , \ p \to \infty \end{cases}$$

Also, the pair $\left(\mathbb{R}^n, \|.\|_p\right)$ is called the l_p-normed or p-normed space. The α-norm is defined by

$$\left\|\overrightarrow{X}\right\|_\alpha := \Delta + (\sec\alpha - \tan\alpha)\delta,$$

where $\Delta = \max\{|x_1|, \ldots, |x_n|\}$, $\delta = \left[\sum_{i=1}^{n} |x_i|\right] - \Delta$, and $\alpha \in \left[0, \frac{\pi}{2}\right)$. The pair $(\mathbb{R}^n, \|.\|_\alpha)$ is called the α-normed space (we refer the reader to [20–22] for a wider treatment).

In Figure 2a, the brown circle represents the unit circle according to the norm l_2 and the red circle represents the circle according to the norm l_p as $p \to \infty$, while the other blue circles represent the circle according to various l_p-norms. Similarly, in Figure 2b, the brown circle represents the circle according to the norm $\alpha = 0$ and the red circle represents the circle according to the α-norm as $\alpha \to \infty$, while the other blue circles represent the circle according to various α-norms. Also, Figure 2 shows that the unit balls are convex with a non-empty interior and centrally symmetric sets according to the l_p and α-norms. However, norms that have unit balls that are centrically symmetric but not mirror symmetric can be defined as follows (see Figure 3):

$$\|(x,y)\|_{2,\infty} := \begin{cases} \|(x,y)\|_2 & , \ x.y \ge 0 \\ \|(x,y)\|_\infty & , \ x.y < 0 \end{cases} \quad \text{and} \quad \|(x,y)\|_{\infty,1} := \begin{cases} \|(x,y)\|_\infty & , \ x.y \ge 0 \\ \|(x,y)\|_1 & , \ x.y < 0 \end{cases}.$$

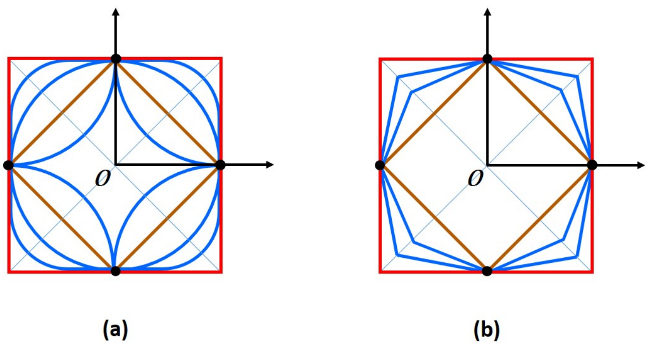

Figure 2. (**a**) Unit circles for varying p-norms, (**b**) unit circles for varying α-norms.

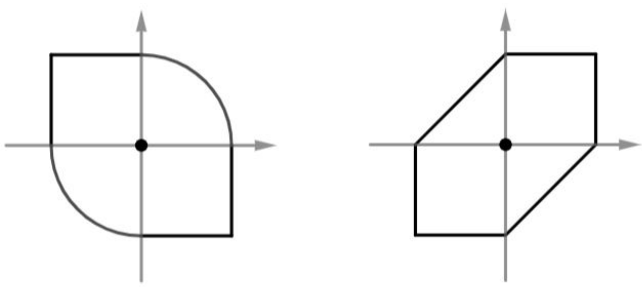

Figure 3. Unit circles in terms of the norms $\|(x,y)\|_{2,\infty}$ and $\|(x,y)\|_{\infty,1}$, respectively.

As a result, it is clear from the definition of a norm that the unit ball has the following characteristics:

(i) The unit ball is a closed and bounded set;

(ii) The unit ball is centrally symmetric;

(iii) The unit ball is convex.

Conversely, if B is a set satisfying the properties **i**, **ii**, and **iii**, then the function

$$\rho_B(P) := \inf\{\lambda \in \mathbb{R}^+ : P/\lambda \in B\}$$

is a norm on $\mathbb{R}^n$, for which B is the unit ball.

It should be noted here that the convex distance function $d_{\rho_D}(Q, P)$ induced by the convex set D,which has a non-empty interior and is not necessarily centrally symmetric, is defined via $\rho_D(P)$ by

$$d_{\rho_D}(Q, P) := \rho_D(P - Q).$$

However, it is obvious that the convex distance function d_{ρ_D} has the properties

$$d_{\rho_D}(\alpha, \beta) = d_{\rho_C}(\alpha - \gamma, \beta - \gamma) \quad (translation\ \ invariance)$$

and $d_{\rho_D}(\lambda\alpha, \lambda\beta) = \lambda d_{\rho_D}(\alpha, \beta)$ (*homogeneity*) such that, for any vectors α, β, γ, and $\lambda \in \mathbb{R}$, the distance function d_{ρ_D} does not have to satisfy the symmetry property, which is one of the axioms of metric space. The distance function d_{ρ_D} satisfies the symmetric axiom if and only if D is centrally symmetric (see [16,17] for more details). Thus, ρ_D is a norm and the distance function d_{ρ_D} is a metric on $\mathbb{R}^n$. The pair $(\mathbb{R}^n, d_{\rho_D})$ is called Minkowski geometry. Thanks to translation invariance and homogeneity, all points in any Minkowski geometry $(\mathbb{R}^n, d_{\rho_D})$ have the same status, and all geometric objects are centered at the origin. Furthermore, the open balls $B_{d_{\rho_D}}(Y, r) = \{X \in \mathbb{R}^n : d_{\rho_D}(Y, X) = r,\ Y \text{ is any point in } \mathbb{R}^n\}$ are rescaled, translated versions of the unit ball $B_{d\rho}(O, 1) = \{X \in \mathbb{R}^n : d\rho(O, X) = 1\}$. As a result, the fact that the norms can be interpreted and understood entirely with the help of the shapes of the unit balls shows that the structure of Minkowski geometry can be completely determined by the unit ball, which is a centrally symmetrical convex body.

The idea of examining the invariance of concepts, including the concept of distance in Euclidean geometry, in Minkowski geometry when the Euclidean norm is replaced by an arbitrary Minkowski norm is quite interesting. However, this trade-off can cause even very elementary and simple questions to be very difficult to answer, and an answer may not even be found. One example is the following question: while in Euclidean geometry the image of a point under circular inversion can be found by a synthetic method due to the unit ball having perfect symmetry, can a synthetic method be constructed for a Minkowski geometry whose unit ball is centrally symmetric but not perfect?

2. Literature Review and Motivation

Although the emergence of inversion is not known for certain in the literature, there are some studies [12,23] that indicate that it was first introduced by Apollonius of Perga (225–190 *BC*). The first systematic research on inversion began with Jakob Steiner (1796–1863), who made many geometric discoveries using this transformation in the 1820s (see [12,23] for details). Later, Mario Pieri [24] developed the subject axiomatically and systematically in the early 1910s. In *What is Mathematics?* [14] by Richard Courant and Herbert Robbins, a detailed description is given of inversion, including Peaucellier's linkage, converting circular into rectilinear motion using the theory of inversion.

When research on inversion is examined, it is seen that the main source is the frequently cited book *Inversion theory and conformal mapping*, written by David E. Blair [25]. Additionally, Gerard A. Venema, in the 10th chapter of his book titled *Fundamentals of Geometry* [26], has presented almost all the properties of circular inversion in a very simple and understandable way. At present, except for the references [27–29], it is very difficult to find documents in the literature containing significant results regarding inversion theory in addition to the results in the essential studies mentioned above. In [27–29], José Luis Ramírez and Gustavo N. Rubiano have introduced the concept of elliptical inversion, which is the generalization of the circular (or spherical) inversion concept, and obtained important results.

In the studies mentioned so far, inversion has been limited to Euclidean geometry. However, the finding of some important but not surprising results regarding inversion in l_1-, α-, and l_∞-normed spaces, respectively, in references [30–32] is evidence that inversion theory is also becoming popular in non-Euclidean geometries.

Especially in the normed spaces $\left(\mathbb{R}^n, \|.\|_p\right)$ and $(\mathbb{R}^n, \|.\|_\alpha)$, an inversion in a circle (or sphere) of radius r centered at the point O is a map of an arbitrary point X found by inverting the length of the displacement vector and multiplying by r^2. If X and X' are inverse points according to this circular inversion, then the circular inversion with respect to the α- and p-norm are defined as follows, respectively:

$$\begin{aligned} X \mapsto X' &= O + r^2 \frac{\overrightarrow{OX}}{\|X-O\|_\alpha^2} \\ X \mapsto X' &= O + r^2 \frac{\overrightarrow{OX}}{\|X-O\|_p^2} \end{aligned} \tag{1}$$

In Euclidean geometry, it can be easily obtained from the definition of inversion that a circular inversion will have the following properties:

i. The inverse of a circle (not through the center of inversion) is a circle;

ii. The inverse of a circle through the center of inversion is a line;

iii. The inverse of a line (not through the center of inversion) is a circle through the center of inversion;

vi. A circle orthogonal to the circle of inversion is its own inverse;

v. A line through the center of inversion is its own inverse;

vi. angles are preserved under inversion.

However, it can be easily seen from the α- and p-circular inversion definitions given in (1) that none of the above-defined properties, except for **v**, remain invariant in the normed spaces $\left(\mathbb{R}^n, \|.\|_p\right)$ and $(\mathbb{R}^n, \|.\|_\alpha)$ (see [31,33] for more details).

Although inversion is a subject that has been known about and studied for a long time, it has been limited to Euclidean logic. It is particularly surprising that inversion theory has hardly been studied within the framework of Minkowski geometry. As far as we know, only references [34–36] can be given as examples of inversion studies in Minkowski geometry. So, there is good motivation for researchers to investigate which geometric features known in Euclidean geometry and related to inversion will remain invariant in Minkowski geometry. In this sense, our main motivation will be to develop a synthetic method to find the inverse of a point under circular inversion in Minkowski geometry. In order to construct this synthetic method, which has not been given before in the literature, in the next section we will take a short journey into circular inversion with the l_1- and l_2-norms in Minkowski geometry.

3. A Visit to Circular Inversion in Minkowski Geometries $(\mathbb{R}^2, l_1)$ and $(\mathbb{R}^2, l_2)$

Let us consider the circle $C = \left\{X \in \mathbb{R}^2 : d(O, X) = r\right\}$. The inverse of an arbitrary point P, different from the center of symmetry of the circle O, with respect to the circle C is the point P', such that

$$d(P, O)d\left(P', O\right) = r^2. \tag{2}$$

A circular inversion is sometimes referred to as a "reflection" relative to the circle. Obviously, if the inverse of point P is point P', then the inverse of point P' is point P. Moreover, from Equation (2), it is clear that, except for the center of the inversion circle O, circular inversion will map the points inside the inversion circle to points outside the circle, or the points inside the circle to points outside. The points on the inversion circle C remain fixed under circular inversion. It should be noted that the center of the inversion circle O cannot be mapped to any point of the plane. However, points close to O are mapped to points far from O, and points far from O are mapped to points close to O. Thus, it would be meaningful to make a mapping between point $\mathbb{R}_\infty$ at infinity and point O under circular inversion. Inversion mapping, which is one of the most important representatives of the conformal functions that generally map circles or lines onto (the same or different) circles or lines and preserves the angles between intersecting curves, is defined as follows (see [25] for more details):

Definition 1. *Let the circle $C = \left\{X \in \mathbb{R}^2 : d(O, X) = r\right\}$ be given. The inversion, according the circle C, is the mapping I_C , defined as follows for all $P \neq O$*

$$\begin{array}{ccccc} I_C^d & : & \mathbb{R}^2 \cup \{\infty\} & \longrightarrow & \mathbb{R}^2 \cup \{\infty\} \\ & & P & \longrightarrow & I_C^d(P) = P' \end{array}$$

such that $I_C^d(P) = P'$ is the point lying on the ray $\overrightarrow{OP}$ and $d(P, O)d(P', O) = r^2$. Additionally, $I_C^d(\mathbb{R}_\infty) = O$ and $I_C^d(O) = \mathbb{R}_\infty$.

According to the above definition, in the plane equipped with the Manhattan norm l_1 (which is given by $\|(x_1, x_2)\|_1 := |x_1| + |x_2|$) and the Euclidean norm l_2 (which is given by $\|(x_1, x_2)\|_2 := \left(|x_1|^2 + |x_2|^2\right)^{1/2}$), circular inversion is formally defined as (see [31,33,37,38] for more details)

$$\begin{array}{ccccc} I_C^d & : & \mathbb{R}^2 \cup \{\infty\} & \longrightarrow & \mathbb{R}^2 \cup \{\infty\} \\ & & P & \longrightarrow & I_C^d(P) = P' = O + r^2 \dfrac{\overrightarrow{OP}}{\|P - O\|_p^2} \end{array}$$

for $p = 1$ and $p = 2$.

According to this definition, it is clear that finding the inverse of a point analytically will lead to tedious operations. In this sense, the following simple synthetic method, which is well known in the literature, has been proposed to find the inverse of a point according to the Euclidean norm l_2.

Simple Construction: *To construct the inverse P' of a point P outside the circle C* (see Figure 4):

Step 1: Draw the tangent lines from point P to inversion circle C;

Step 2: Label the point where the tangent lines touch circle C as T_1 and T_2;

Step 3: The point where line segment $\overline{T_1T_2}$ intersects the ray $\overrightarrow{OP}$ is the inverse of P.

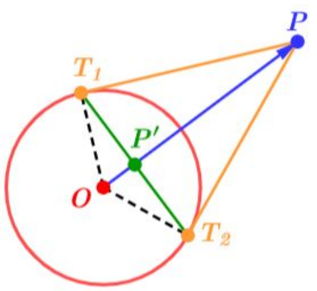

Figure 4. A standard construction for Euclidean circular inversion.

The inverse of a point in Euclidean geometry can be easily found by the above well-known synthetic method since the unit ball has perfect symmetry. However, a synthetic method for Minkowski geometry, in which the unit ball is centrally symmetric but does not necessarily have perfect symmetry, is not available in the literature. In order to create a synthetic method for circular inversion in Minkowski geometry, we will consider p- and α-normed spaces, which are the most classical examples of Minkowski geometry. If we manage to generate a synthetic method for Manhattan (or taxicab) geometry, which corresponds to the special case of $p = 1$ or $\alpha = \pi/2$ in p- and α-normed spaces, we can attempt to generalize this method to Minkowski geometry. Thus, the method we will give for Manhattan geometry in the next section will illuminate our way forward in this difficult journey regarding the synthetic method we want to give for Minkowski geometry. Here, we should immediately point out that the effort to develop a synthetic method for circular inversion in Minkowski geometry is a motivation for our future work. In this study, we will be content with only the norm l_1.

4. Main Result

4.1. Notation and Preliminary

We will denote the metric induced by the l_1-norm as d_T, which is called the taxicab or Manhattan metric, and formulate it as $d_T(P_1, P_2) = \left\|\overrightarrow{P_1P_2}\right\|_1$. Similarly, the metric induced by the l_2-norm is formulated as $d_E(P_1, P_2) = \left\|\overrightarrow{P_1P_2}\right\|_2$. The similarity of triangles will be denoted by "$\sim$". Also, from the definition of the metric d_T, the following proposition is quite clear (see [39] for more details).

Lemma 2. *Let L be a line through the points P_1 and P_2 in the analytical plane. If L has slope m, and is not a vertical line,*

$$d_E(P_1, P_2) = \left(\frac{\sqrt{1+m^2}}{1+|m|}\right) d_T(P_1, P_2).$$

If L is a vertical line, then $d_E(P_1, P_2) = d_T(P_1, P_2)$.

4.2. Synthetic Construction of Circular Inversion in l_1-Normed Space

Let C_T be an l_1-normed circle with the radius r_T and center O.

If P is an exterior point to C_T, then the point Q, which is the intersection of the circle C_T by the ray $\overrightarrow{OP}$ is determined. Afterward, an Euclidean circle C is constructed with the radius $d_E(O, Q)$ and centered at the point O. Let the points where the tangent lines drawn from the point P to the Euclidean circle touch the Euclidean circle be labeled T_1 and T_2. Thus, the point P' where the line segment $\overline{T_1T_2}$ intersects the ray $\overrightarrow{OP}$ is the inverse of the desired point P (see Figure 5).

Indeed, it is simply observed that $\triangle OPT_1 \sim \triangle OT_1P'$, hence

$$\frac{d_E(O, P)}{d_E(O, T_1)} = \frac{d_E(O, T_1)}{d_E(O, P')}.$$

Therefore,

$$d_E(O, P) d_E(O, P') = (d_E(O, T_1))^2 = (d_E(O, Q))^2.$$

Using the relation given in Lemma 2, which allows conversion between the Euclidean metric d_E and the taxicab metric d_T, we obtain that

$$\left(\frac{\sqrt{1+m^2}}{1+|m|}\right) d_T(O, P) \cdot \left(\frac{\sqrt{1+m^2}}{1+|m|}\right) d_T(O, P') = \left(\left(\frac{\sqrt{1+m^2}}{1+|m|}\right) d_T(O, Q)\right)^2.$$

By making simplifications to the above equation, we find that

$$d_T(O, P) d_T(O, P') = (d_T(O, Q))^2 = (r_T)^2.$$

If point P is taken outside the circle, only the roles of points P and P' will change. Therefore, the synthetic construction is the same.

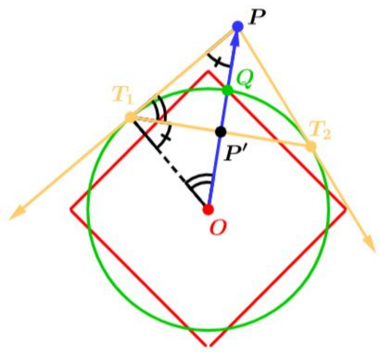

Figure 5. A new construction for circular inversion in terms of the l_1-norm.

Theorem 1 (Main Theorem)**.** *Let C_T be a l_1-normed circle with the radius r_T and center O in the Minkowski plane $(\mathbb{R}^2, l_1)$. Then, the image of a point P under inversion with respect to C_T is*

$$P^{\shortmid} = \overrightarrow{OP} \cap \overline{T_1T_2},$$

where T_1 and T_2 are the points made by the tangents drawn from point P to a Euclidean circle with radius $r = d_E(O, Q)$, such that $Q = C_T \cap \overrightarrow{OP}$ and the center at O touch the Euclidean circle.

5. Conclusions

Open problem: In Minkowski geometry with an l_2-norm (Euclidean geometry), the image of a point under circular inversion can be found by a synthetic method. Similarly, can a synthetic method be constructed to find the image of a point under circular inversion in Minkowski geometry (geometry that emerges through norms in a finite-dimensional Banach space)?

The fact that circles corresponding to the norm in Minkowski geometry are centrally symmetric but do not always have mirror symmetry makes this question quite difficult. In this work, we have succeeded in providing a synthetic method for circular inversion in l_1-normed spaces. Based on the method given here, we will try to generalize this method to Minkowski geometry, regardless of the norm, in our future work.

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