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PHI-BONACCI IN ANCIENT GREECE

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Abstract: Fibonacci numbers are a very popular subject in mathematics, culture and science. A major open question is why the Ancient Greek overlooked this series, while they were very familiar with the golden mean and division in extreme and mean ratio. Furthermore, they could compute the square root of five to any precision using Theon's ladder. This is based on tables built with side and diagonal numbers, and it is a very efficient method to compute roots of integers, methods of incredible simplicity and efficiency, which are little known even now to most of the experts. The biologist D'Arcy Wentworth Thompson showed that the same method could be used to generate the Fibonacci series, when a simple shift in the computation of the tables is used. He argues, quite convincingly, that the Greek could not have overlooked this. Actually, the same method can be used to generate all possible regular phyllotaxis patterns.

Keywords: Fibonacci, square roots, Theon, Theodorus

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FIBONACCI NUMBERS IN SCIENCE

Fibonacci numbers are a very popular subject of research and recreation and one can find innumerable articles in mathematics, science, architecture and the arts. Especially in the latter fields they have achieved an almost divine status, because of the relation to the golden mean. From a scientific point of view however, one has to be very cautious in the application of the series to actual natural or cultural phenomena. For example, in the arrangement of leaves the Fibonacci numbers relate the number of spirals going in one direction, to the number of spirals in the other. In a large-scale experiment of popular science, with over 600 sunflowers only 3 out of 4 of the parastichies on sunflowers were direct Fibonacci numbers. The other 1/4 (i.e. 25%) were approximate or modified Fibonacci and Lucas numbers, derived series, or irregular (Swinton et al., 2016). One can contrast this with models whose applicability is 100%. For example, in square bamboos all possible cross sections can be described uniquely, with superellipses and supercircles or Lamé curves (Huang et al., 2020).

The French mathematician Gabriel Lamé (1795-1860) did extensive work on the recurrent series $u_{n+2} = u_{n+1} + u_n$ with initial conditions $u_0 = 0; u_1 = 1$, and for this reason it was known as the Lamé series (Lucas, 1876). Despite the fact that various mathematicians had worked on this series, it was apparently only in 1877 that the name Fibonacci was linked to this recurrent series (Lucas, 1877). Lamé's work with the series was purely mathematical, in contrast to superellipses, which he developed to deal with shapes of natural shapes, in particular crystals. Two centuries later superellipses are used to model leaves, seeds, tree rings and bamboo stems. While they turn out to be excellent scientific models for a large class of natural shapes with fourfold symmetry, one would not expect superellipses to achieve the same success rate with pentagonal or triangular shapes. In the very same way Fibonacci series cannot be implemented as the ultimate model for phyllotaxy, or for understanding ancient architecture or art.

In his wonderful book On Growth and Form D'Arcy Thompson (1917) deals with application of geometry in the growth and form of biological objects, linking certain forms to findings in mathematics and mathematical physics. He devotes a full chapter to phyllotaxis and relation to Fibonacci numbers, in effect taking away the magic or mysticism surrounding this particular series. On the other hand, Jean (2009) is essential reading for student of plant phyllotaxis.

Since leaves and scales on pinecones are discrete structures one should look to difference equations, polynomials and logarithmic spirals to study phyllotaxy (Gielis et al., 2020). For example, Chebyshev polynomials, Lucas L_n and Fibonacci numbers F_n can all be considered as special cases of the homogeneous linear second order difference equation with constant coefficients $u_0;\ u_1;\ u_{n+1} = au_n + bu_{n-1},$ for $n \leq 1$. If a and b are polynomials in x, a sequence of polynomials is generated. In particular if $a = 2x$ and $b = -1$, we obtain Chebyshev polynomials. They are of the first kind $T_n\,(x)$for $u_0 = 1;\ u_1 = x$, and of the second kind $U_n\,(x)$ for $u_0 = 1;\ u_1 = 2x$. Fibonacci numbers F_n arise for $a = b = 1;\ u_0 = 0;\ u_1 = 1$. For $a = b = 1;\ u_0 = 2;\ u_1 = 1$, we obtain Lucas numbers L_n. Therefore, if in Chebyshev polynomials $i = \sqrt{-1}$ is used with $x = \frac{i}{2}$ the results are Lucas numbers L_n for Chebyshev polynomials of the first kind T_n, and Fibonacci numbers F_n for those of the second kind U_n (Gielis et al., 2017). There is a range of other beautiful connections (Ricci, 2020).

The true origins of the Fibonacci series are shrouded in the mists of time; it was known well before Fibonacci in India (Lucas, 1877; Singh, 1985). Since this series is about numbers, an obvious question arises: Why did the Ancient Greeks, whose contributions in mathematics and science provide for the foundations of our current science, fail to come up with this series? They knew the golden section and had a great interest in the decagon, the pentagon and their related solids. Euclid IV. 10 states that the side of the decagon is equal (in terms of the radius) to the Golden Mean $= \frac{\sqrt{5}-1}{2} =$ 0,618….. Moreover, they had various arithmetical methods for easy computations of roots, inherited from Egyptians and Babylonians. So, why did they fail to discover the Fibonacci numbers?

SIDE AND DIAGONAL TABLES

It is a question that D'Arcy Thompson asked himself almost 100 years ago; and answered in his brilliant style. In *On Growth and Form* D'Arcy Thompson has a small footnote in the section On leaf arrangement or phyllotaxis, which links to a paper of his own, in the journal *Mind* (D'Arcy Thompson, 1928). In his mind, it is simply inconceivable that the Greek overlooked this series. He develops some very convincing arguments for his hypothesis and does so in the same brilliant literary style as in his book. Now D'Arcy Thompson was not just some biologist;

he translated the *Historia Animalium* of Aristotle in 1910, and before that, he wrote *A glossary of Greek birds* on all birds found in Greek literature and writings. D'Arcy Thompson's father was a professor of Greek.

The article in *Mind* is called "*Excess and defect: or the little more and the little less*" and he starts out with the following question on what Aristotle had in mind when defining a number:

"Aristotle gives us the following statement of Plato's concept of the "genesis of number": Number is derived from Unity and the indeterminate dyad (Metaphysics 1081a, 15); but this apparently simple statement has never been satisfactorily explained. Though we do our best to collate it with other related passages we are left in doubt in the end; there is confusion or contradiction somewhere, which no man has found his way through; and I begin to think that our first business is to enquire what is meant by "number" in this particular connection. Aristotle's statement might refer, and it is usually supposed to refer, to the genesis of Number in its widest sense, to the genesis of the ordinary numbers 1, 2, 3 . . . from one and from one another: a question which is either simplicity itself or a transcendental problem of extreme subtlety. This particular process of generation has never been shown to be related to the so-called indeterminate dyad (ἀόριστος δυάς).

On the other hand, arithmos (ὁ ἀριθμός) may be used here in its technical sense, meaning a surd or "irrational number," especially $\sqrt{2}$; and the general problem of Number may never have been in question at all. It was the irrational number, the numerical ratio (if any) between two incommensurable segments, which was a constant object of search, whose nature at a number was continually in question, and whose genesis as a number cried aloud for explanation or justification. I am inclined to think that this restricted but vitally important problem is the question at issue; but if it be only part of a more general question, it is still the only part thereof, which seems capable of explanation. In short, if we keep to this restricted definition of our problem, and if we then go a step or two farther in its interpretation than Prof. Taylor has gone, we come to a very simple understanding of what the One (τὸ ἕν) and the indeterminate dyad (ἀόριστος δυάς) are; and of how, between them both, such a "number" as $\sqrt{2}$ is generated.

The ' side and diagonal numbers,' as Theon and Iamblichus explain them, hark back to the all-important Theorem of Pythagoras, and to the simplest case thereof where the right-angled triangle is also isosceles. By their means we 'arithmetize' this construction, and for certain values of the 'side' obtain 'rational values' for the corresponding diagonal; consequently, dividing the diagonal-number by the side-number, we obtain an approximation to or a 'rational value' for $\sqrt{2}$, the true ratio of diagonal to side. It is part of the great Pythagorean principle of letting Mathematics rest on an arithmetical basis.

It is just worth mentioning that what we here call the diagonal is called in Greek the diameter; it is the diagonal of the completed square (or parallelogram), and the diameter of the circle in which it can be inscribed.

The following is a table of the side and diagonal numbers *(*πλευρικοὶ καὶ διάμετρικοὶ ἀριθμοί*).* Proclus gives the series as far as 12,17, and adds:

Sides (**πλευραί**)	**Diagonals** *(ΔΙΆΜΕΤΡΟὶ)*
1	1
2	3
5	7
12	17
29	41
70	99
169	239

We begin, necessarily, with 1, as the origin (ἀρχή) of both series; for, as Theon says, Unity is the first principle of all configurations, and consequently there is in Unity a 'logos' (λόγος) both of diagonal and of side. If the side of the triangle measures One, One must represent the diagonal also, as its nearest rational number or equivalent. The further construction of the table may be described in various ways, according to its various properties. The simplest way, perhaps, is to say that we add a side-number to its corresponding diagonal to get the next side-number ($2 + 3 = 5$); and a side-number to its immediate predecessor to get the next diagonal ($5 + 2 = 7$), etc. We may also say that each number, whether side or diagonal, is equal to twice its immediate predecessor plus the one before that $s_n = 2s_{n-1} + s_{n-2}$, etc………

……The table of side and diagonal numbers has many other properties. For instance, as Proclus tells us, the sum of the squares of two adjacent diagonals = twice the sum of the squares on the two corresponding sides, e.g. $3^2 + 7^2 = 2(2^2 + 5^2)$. And, in Chapter xxiii he shows, following Adrastus, that the sum of the squares of 'all' the diagonals is equal to twice the sum of the squares of ' all' the sides.

As Prof. Taylor explains, this table is precisely equivalent to what, in our arithmetic, we call a Continued Fraction, viz.,

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{etc}}}}$$

But while we may illustrate our problem in this way, I do not think we simplify it. The continued fraction is an elegant arithmetical device, and the mathematician calls it a simplified expression; but it does not follow that it is simple to work with. Carry it on to ten or twenty terms, and it becomes a troublesome matter to evaluate; while the Greek side-and- diagonal numbers may be carried as far as we please, and still require only the easiest arithmetic.

The Greek table has another advantage over our continued fraction, in that it obviously is just what it purports to be, namely an arithmetization of the corresponding geometrical figure. We have merely to take twice the square of a side number to get, approximately, the square of the opposite diagonal number: and when we proceed to do so systematically, we discover three curious and important things.

Firstly, the successive results are closer and closer approximations to that irrational number (viz. $\sqrt{2}$) which is the ' limit,' the un-attainable limit, of the series. Secondly, the approximations are alternately on one side or the other, a little more or a little less than the number at which we aim; and herein lies, as Prof. Taylor explains, the technical meaning in Greek arithmetic of 'excess and defect'. Thirdly, the striking and beautiful fact appears that this 'excess or defect' is always (in this case) capable of being expressed by a difference of 1. The square of the diagonal number {i.e., of what Socrates calls the ' rational diagonal') is alternately less or more by one than the sum of the squares of the sides:

$$2 \cdot 1^2 = 1^2 + 1,\ 2 \cdot 2^2 = 3^2 - 1, 2 \cdot 5^2 = 7^2 + 1,\ 2 \cdot 12^2 = 17^2 - 1 \quad \text{etc.}$$

This property of the side-and-diagonal series, that not merely is the square of the one in alternate excess and defect as compared with twice the square on the other, but that this alternate excess and defect is in every case measured by one unit, is expressly stated by Theon and by Proclus.

Similar tables can be constructed, as the Greeks well knew, for other square roots; and the way to construct them is in each case easy to discover. For instance, the table for $\sqrt{5}$ is as follows:

1	2
4	9
17	38
72	161
etc	

According to which table,

$5 \cdot 1^2 = 2^2 + 1;\ 5 \cdot 4^2 = 9^2 - 1;\ 5 \cdot 17^2 = 38^2 + 1;\ 5 \cdot 72^2 = 161^2 - 1$, etc

The table for $\sqrt{17}$, to which allusion is made in the Theaetetus, would run,

1	4
8	33
65	etc

Hence

$17 \cdot 1^2 = 4^2 + 1; 17 \cdot 8^2 = 33^2 - 1$

Observe how the 'One' comes in, to 'equalize' all of these".

THEON'S LADDER

Theon of Smyrna lived in the second century AD. The simple method to compute roots, is known as Theon's Ladder. He was a neo-Pythagorean and wrote a book on the necessary mathematics to understand Plato's writing. At the beginning of the 20th century there was a great interest in these books, and 'Prof. Taylor' referred to above, was one of the leading authorities (Taylor, 1926a,b). D'Arcy Thompson's article was inspired by his earlier articles and book.

FIGURE 1 SHOWS THE SPIRAL OF THEODORUS, *THE GEOMETRIC CONSTRUCTION USING RECTANGULAR TRIANGLES, OF WHICH ONE SIDE IS EQUAL TO 1. THEODORUS OF CYRENE (CA. 466–399 B.C.) IS THE TEACHER OF PLATO AND THEAETETUS AND IS CREDITED WITH THE PROOF OF THE IRRATIONALITY OF* $\sqrt{N}$, $N = 2,3,5,\ldots,17$ *(GAUTSCHI, 2009). ALSO, IN THE SPIRAL THE ONE IS A CRUCIALLY IMPORTANT NUMBER, AS A DEFINITION TO WHICH ALL REST MUST BE MEASURED. IT ALSO SERVES AS EQUALIZER. HE MAY HAVE KNOWN ALSO OF A DISCRETE SPIRAL, TODAY NAMED AFTER HIM, WHOSE CONSTRUCTION IS BASED ON THE SQUARE ROOTS OF THE NUMBERS.*

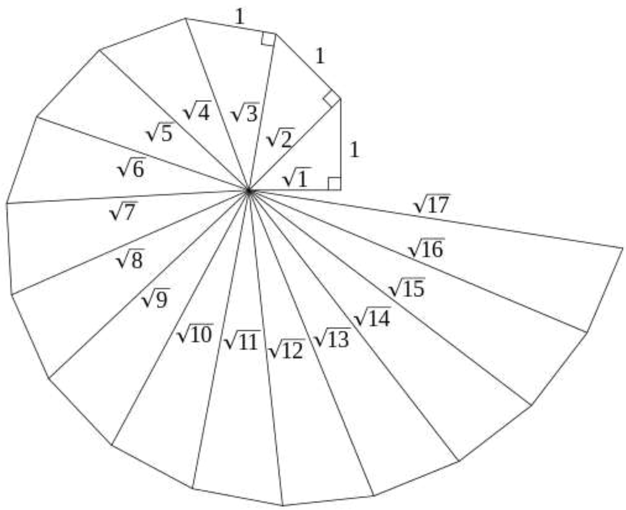

Figure 1: The spiral of Theodorus of Cyrene

It shows up in these beautiful methods in relation to the square roots of 2, 5 and 17. The side and diagonal numbers are also found in the Online Encyclopedia or Integer Numbers OEIS (Sloane and Plouffe, 1995). The side numbers in the table for $\sqrt{2}$ are <u>A1000129</u>, the Pell numbers. The diagonal numbers for $\sqrt{2}$ are A1001333, "the denominators of continued fractions for $\sqrt{2}$." A001076 and A001077 are "the numerators and denominators of continued fractions for $\sqrt{5}$", respectively, and they are also the side and diagonal numbers for the tables of

$\sqrt{5}$. But since it works for square roots of all integers, OEIS should be updated with this knowledge.

COMPUTING ROOTS OF INTEGERS

About the tables, D'Arcy Thompson writes that "*the way to construct them is in each case easy to discover*". This is indeed the case and can in fact be done for any integer, not only the ones given above. The general rule for building these tables is always the same. The numbers s_n denote the side numbers in the left column of the side-diagonal table, and the numbers d_n denote the diagonal numbers in the right column, and the rule is:

$$\mathrm{s_n} = \mathrm{B} \cdot \mathrm{s_{n-1}} + \mathrm{d_{n-1}}$$

$$\mathrm{d_n} = \mathrm{B} \cdot \mathrm{s_n} + \mathrm{s_{n-1}}$$

where B is the multiplier. For $\sqrt{2}$ it is 1, for $\sqrt{5}$ it is 2, and for $\sqrt{17}$ it is 4. This corresponds to the square of the base number B just one lower than the number over which the square root is taken. For example, $5 = 2^2 + 1$ and $17 = 4^2 + 1$, so $B = 1$ and $B = 4$ for 5 and 17, respectively. As D'Arcy Thompson states, this is a very efficient method for finding roots, easy to compute.

If we expand the table of side and diagonal numbers to eight rows, and with the ratio: $\frac{s_n}{s_{n-1}}$, (i.e. $\frac{s_8}{s_7}$), we arrive at a certain number. If from these ratio's the base number **B** is subtracted, i.e. $(\frac{s_n}{s_{n-1}} - \boldsymbol{B})$**,** we arrive at the required roots.

NUMBER	2	5	17
$\frac{s_n}{s_{n-1}}$	2,4142011834	4,2360679700	8,1231054863
$\frac{s_n}{s_{n-1}} - B$	1,4142011834	2,2360679700	4,1231054863

If we compute now the difference between the "real value" to 10 decimals this gives very low differences. In the table below this is computer when the side-diagonal table has 8 and 10 rows.

	TABLE ROWS	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{17}$
COMPUTED VALUE CV		1,4142135624	2,2360679700	4,1231056256

DELTA (CV, $\frac{s_n}{s_{n-1}} - B$)	8	0,0000123789	0,0000000075	0,0000001393
DELTA (CV, $\frac{s_n}{s_{n-1}} - B$)	10	0,0000003644	0,0000000000	0,0000000000

This is very simple and general. It can be done for any integer. The base number (or first diagonal number) is simply the square root of the previous number. Hence, to compute if the $\sqrt{3}$ the first side number is again one, but the first diagonal number is $\sqrt{\mathbf{2}}$ =1,4142011834.

The procedure was probably so well-known and simple, that one had to wait until the 4th century before Theon wrote it down. It is not widely known in our days, but it must have been known to a larger audience one century ago, with Taylor and Thompson. Not only can this be done for any integer but also for (at least in principle) for any real number; one can remove the comma of decimal system and put it back after the operations. Only quite recently, it was shown that Theon's original method is naturally generalized for the calculation of any root, $\sqrt[n]{c}$, where $1 < c$ (Giberson and Osler, 2004; Osler et al, 2008).

THE FIBONACCI NUMBERS

Remember that the real challenge D'Arcy Thompson wants to address is that for him it is inconceivable that the Greek had overlooked the Fibonacci numbers. He found that the side-diagonal tables used to compute roots, also directly lead to Fibonacci numbers, if only one small change is made in construction the tables. Again, we let D'Arcy Thompson speak:

"THERE IS STILL ANOTHER TABLE WHICH MAY BE JUST AS EASILY, OR INDEED STILL MORE EASILY DERIVED FROM THE FIRST, AND WHICH IS OF VERY GREAT IMPORTANCE. YET THERE IS NO ACCOUNT OF IT, NOR THE LEAST ALLUSION TO IT, IN ALL THE HISTORY OF GREEK MATHEMATICS; AND IT IS COMMONLY BELIEVED TO HAVE BEEN FIRST MADE KNOWN BY THE GREAT ARITHMETICIAN WHO INTRODUCED THE ARABIC NUMERALS INTO THE CHRISTIAN WORLD. WE REMEMBER THAT, TO FORM OUR TABLE OF SIDE AND DIAGONAL NUMBERS, WE ADDED EACH SIDE-NUMBER TO ITS OWN PREDECESSOR, THAT IS TO SAY, TO THE NUMBER STANDING IMMEDIATELY OVER IT IN THE TABLE, AND SO WE OBTAINED THE NEXT DIAGONAL; THUS, WE ADD 5 TO 2 TO GET 7, IN THE FOLLOWING:

1	**1**

2	3
5	7
12	etc

But suppose that, instead of adding 5 to 2, to make 7, we should add 5 to 3, and make 8: it is just as easy, and seems just as natural. In other words, suppose we keep on adding each side-number to the preceding diagonal, —that is to say, to the number which stands obliquely instead of vertically above.

We then get the following table:

1	**1**
2	3
5	8
13	21
34	55

This is the famous series, sometimes called the Fibonacci series, supposed to have been 'discovered' or first recorded by Leonardo of Pisa, nicknamed the Son of the Buffalo, or "Fi Bonacci". This series has more points of interest than we can even touch upon. It is the simplest of all additive series, for each number is merely the sum of its two predecessors. It has no longer anything to do with sides or diagonals, and indeed we need no longer write it in columns, but in a single series, 1, 1, 2, 3, 5, 8, 13, 21, etc. It is identical with the simplest of all continued fractions,

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\text{etc}}}}$$

Its successive pairs of numbers, or fractions, as $\frac{5}{3}, \frac{8}{5}$ etc., are familiar to botanists, ever since Bravais showed them to express the number of spirals which may be counted to right and to left, on a fir-cone or any other complicated inflorescence.

Here is another of the many curious properties of the series: $0^2 + 2^2 = 2(1^2 + 1^2)$; $1^2 + 3^2 = 2(1^2 + 2^2)$; $1^2 + 5^2 = 2(2^2 + 3^2)$; $2^2 + 8^2 = 2(3^2 + 5^2)$, etc.

But the main property, the essential characteristic, of these pairs of numbers, or fractions, is that they approximate rapidly, and by alternate excess and defect, to the value of the Golden Mean, that is to say to the value of $\frac{\sqrt{5}-1}{2} =$ 0,618. Thus, the successive fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}$ etc, expressed in decimals, are as follows: 0.5; 0.66. . ;5….. 0.6190 . ., 0.6176 . ., 0.6181 . ., etc.

The Golden Mean itself is, of course, only the numerical equivalent, the 'arithmetisation,' of Euclid II. 11; where we are shown how to divide a line in "extreme and mean ratio," as a preliminary to the construction of a regular pentagon: that again being the half-way house to the final triumph, perhaps the ultimate aim, of Euclidian or Pythagorean geometry, the construction of the regular dodecahedron, Plato's symbol of the Cosmos itself.

……And in our table, any three consecutive numbers may represent these three geometrical magnitudes, the square of the intermediate number being equivalent—approximately equivalent—to the product of the other two. Observe that, precisely as in the former case, the approximation gets closer and closer; there is alternate excess and defect; and (above all) the "One" is needed in every case, to equate the terms, or remedy the defective approximation.

$5^2 = 3 \cdot 8 + 1$; $8^2 = 5 \cdot 13 - 1$; $13^2 = 8 \cdot 21 + 1$; ETC".

Following Thompson, if we consider how tables are constructed, there is only and only one simple difference, namely the computation of d_n, where for roots $d_n = s_n + s_{n-1}$, but to get Fibonacci numbers the computation of d_n, the previous diagonal number d_{n-1} is used instead of the previous side number s_{n-1}. The table below shows that this involves only one cell shifted to the right:

$\sqrt{2}, \sqrt{5}, \sqrt{17}$	s_{n-1}	d_{n-1}	*B* . light green + olive green gives dark green = s_n (with *B* the same multiplier as above)
	s_n		
	s_{n-1}		*B* . light green + olive green gives dark green = d_n
	s_n	d_n	
Fibonacci[1]	s_{n-1}	d_{n-1}	Light green + olive green gives dark green = s_n
	s_n		
		d_{n-1}	Olive green + light green gives dark green = d_n
	s_n	d_n	

These then are the arguments set forth in D'Arcy Thompson's gem article. The Golden Mean $= \frac{\sqrt{5}-1}{2} = 0{,}618.....$; or the Section (τομή) as the Greeks called it, can be computed in easy way, to any degree of precision. For D'Arcy Thompson, with his knowledge of Greek language and his insight into arithmetic and geometry, there could be little doubt:

"IT IS QUITE INCONCEIVABLE THAT THE GREEKS SHOULD HAVE BEEN UNACQUAINTED WITH SO SIMPLE, SO INTERESTING AND SO IMPORTANT A SERIES; SO CLOSELY CONNECTED WITH, SO SIMILAR IN ITS PROPERTIES TO, THAT TABLE OF SIDE AND DIAGONAL NUMBERS WHICH THEY KNEW FAMILIARLY. BETWEEN THEM THEY "ARITHMETICIZE" WHAT IS ADMITTEDLY THE GREATEST THEOREM, AND WHAT IS PROBABLY THE MOST IMPORTANT CONSTRUCTION, IN ALL GREEK GEOMETRY. BOTH OF THEM HARK BACK TO THEMES WHICH WERE THE CHIEF TOPICS OF DISCUSSION AMONG PYTHAGOREAN MATHEMATICIANS FROM THE DAYS OF THE MASTER HIMSELF; AND BOTH ALIKE ARE BASED ON THE ARITHMETIC OF FRACTIONS, WITH WHICH THE EARLY EGYPTIAN MATHEMATICIANS, AND DOUBTLESS THE GREEK ALSO, WERE ESPECIALLY FAMILIAR. Depend upon it, the series which has its limit in the Golden Mean was just as familiar to them as that other series whose limit is $\sqrt{2}$.

The Golden Mean series is a very curious one; and as we have put it, it is only in one, and that the simplest, of its many forms. For the fact is, we may begin it as we please, with 1, 1, or 1, 2, or 1, 3, or any two numbers whatsoever, whole or fractional, and in the end, it comes always to the same thing. For instance, we may have the series

1, 5, 6, 11, 17, 28,45, 73, 118, 191, 309, etc.,

which only agrees with the former in that each number is the sum of its two predecessors: but as before, the fractions soon approximate closely to the Golden' Mean; 191/309 = 0.61812 . . . ; and (as a consequence) 309/191 = 1.618 . . . approximately. These two methods, of finding the value of $\sqrt{2}$ and the value of the Golden Mean, are, be it remarked, by no means mere rough approximations, but they actually lead, more easily and quickly than does our modern arithmetic, to results of extreme accuracy. In the case of the side and diagonal numbers we need go no farther than the tenth place in the table (as can be done in less than two minutes) to get a fraction which is equivalent to the value of $\sqrt{2}$ to six places of decimals!"

The rules of combining side and diagonal numbers as proposed by D'Arcy Thompson is simply addition of consecutive terms in a recurrent series. As Thompson writes: "For the fact is, we may begin it as we please, with 1, 1, or 1, 2, or 1, 3, or any two numbers whatsoever, whole or fractional, and in the end, it comes always to the same thing". In the case of Fibonacci numbers, the first diagonal number is 1, but if the first diagonal number is 3 and the table is constructed in the same way as the Fibonacci numbers (i.e. $s_n = s_{n-1} + d_{n-1}$ and $d_n = d_{n-1} + s_n$) then one obtains the Lucas numbers.

1	**3**
4	7
11	18
29	47
76	123, etc

This method can be used to generate the phyllotactic patterns observed in plants (see Zagórska-Marek, 1995; Figure 1). If this pair is (1,1) or (1,2) one obtains the Fibonacci series, the so-called main sequence. In the accessory series, the first side number is 1. Then the first pair of side and diagonal numbers (1,3) will generate the numbers Lucas series (1,3,4,7,..), also known as the first accessory series. The second and third accessory series starts with (1,4) and (1,5).

The so-called multijiugate main sequence starts with (2,4) leading to every term in the Fibonacci series doubled, namely $2 \cdot (1,1,2,3,5,..)$. In the bijiugate first accessory series the sequence is double of the Lucas series $2 \cdot (1,3,4,7,..)$. And so on. Finally, the lateral sequences start with the side number 2 and for first diagonal number odd numbers ≥ 5 are used (using 3 generates again the Fibonacci series). This lateral sequence is also known as the anomalous phyllotaxis (Jean, 2008).

Φ IN ANCIENT GREECE AND THE GRAND VISION OF D'ARCY THOMPSON

The Phi in the Phi-bonacci of the title is the Φ (Phi) which was well known in Ancient Greece, e.g. with the sculptor Phidias. The Golden Mean or the Section was well known to mathematicians, and they could compute it from the side-diagonal table for $\sqrt{5}$. In D'Arcy Thompson's opinion, they hardly could have overlooked tables, which lead directly to Φ.

This article is also a tribute to Sir D'Arcy Thompson, a great scientist. His magnum opus On Growth and Form is a true classic, but the paper in Mind also testifies of his diligence and devotion to understand, and of his creative mind. This stands in stark contrast to the many biologists who have dismissed this wonderful book for a variety of reasons; not in the least because he did not discuss genetics or Darwin. But that was exactly the point; he wrote the book as a counterweight to the emphasis of biology on the hereditary aspects. For him the shapes that are found in nature are the result of forces, whatever their nature, and the interplay between the organism and the environment.

This is what the natural sciences and natural philosophy, from the Ancient Greek up to Newton, are all about a certain sense his vision was grander than Darwin's, since Charles Darwin only dealt with the living. In his book, D'Arcy Thompson also characterized himself when writing: "The search for differences or fundamental contrasts between the phenomena of organic or inorganic, of animate or inanimate things, has occupied many men's minds, while the search for community of principles or essential similitudes has been pursued by few".

He was also very knowledgeable about Greek science and literature, like many learned men of the past (Thompson, 1929b).

His book is really about this grand vision: “So the living and the dead, things animate and inanimate, we dwellers in the world and the world in which we dwell - πάντα γα μὰν τὰ γιγνωσκόμενα - are bound alike by physical and mathematical law”. The Greek translates as “Everything we can know”, living and non-living.

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