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ParabolicTrigonometry

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ABSTRACT

Different forms of trigonometry have been proposed in the past to account for geometrical and applicative issues. Along with circular trigonometry, its hyperbolic counterpart has played a pivotal role to provide the geometrical framework of special relativity. The parabolic trigonometry is in between the previous two, we discuss the relevant properties, point out the analogies with the standard forms and discuss perspective applications

1 Introduction

Circular trigonometry is a common background of high school pupils, the hyperbolic counterpart is less known and its relevance to special relativity [1] is limited to the university courses. The elliptic functions are not widespread known as they should. A non-superficial understanding of their properties requires more advanced knowledge of calculus [2].

The concepts associated with trigonometry and with its generalization can be framed within different contexts. The algebraic point of view allows the possibility of introducing the trigonometric functions as being associated with the exponenziation of generalized imaginary units [3] and opens the way to higher order ***"trigos"*** in the sense discussed in ref. [4]. An alternative program foresees definitions extending to higher powers the Pythagorean identity of ordinarytrigonometric functions, such a point of view identifies new trigos, with their own geometricalinterpretation on elliptic curves and with different numbers playing the role of π [5].

We will exploit the last method to study a class of trigonometric functions in between the circular and hyperbolic cases.
The trigonometric parabolic functions (TPF) ${}_pc(\phi),\ {}_ps(\phi)$(parabolic cosine and sine respectively) are defined in fig. 1 as the coordinates of the point *P*, taken on the parabola of equation $y=1-x^2, -1\le x\le 1$, which defines (see below for further comments) the Fundamental Parabola (FP-I).

It is evident that the PTF's satisfy the "fundamental" Parabolic-Pythagorean-identity [6,7]

$$ {}_pc(\phi)^2+{}_ps(\phi)=1 \qquad (1)$$

where ϕ is twice the area sector, provided by the dashed region in Fig.1. It plays the same role of the angle for the circular trigonometric functions and the PTF can be written in parametric form by exploiting it as reference parameter.

According to the definition of area sector and with reference to Fig. 1, we get the following identity providing a link between PTF and ϕ

$$\frac{1}{2}\,{}_pc(\phi)\,{}_ps(\phi)+\int_{{}_pc(\phi)}^{1}(1-\xi^2)d\xi=\frac{1}{2}\phi \qquad (3)$$

which will be exploited to study the relevant properties under derivatives.

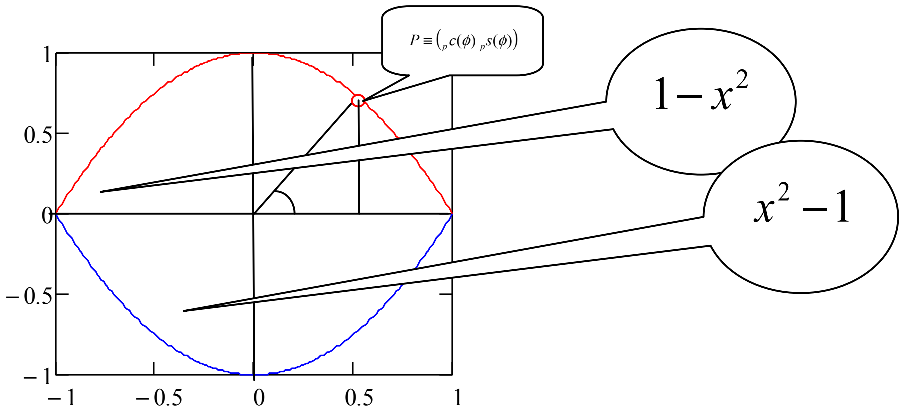

Fig. 1

Definition of the "Parabolic-Trigonometric-Functions" as the coordinates of the point $P\equiv\left({}_pc(\phi)\;{}_ps(\phi)\right)$ ***on the parabola of equation*** $y=1-x^2$***. The angle*** φ ***isthe circular angle corresponding to the area sector*** ϕ***.***

Keeping the derivative of both sides of (3) with respect to ϕ and using eq. (1), we end up with the following equation, specifying the properties under derivative of PTF[1]

$$\begin{aligned}&\frac{d}{d\phi}{}_pc(\phi)=-\frac{1}{1+{}_pc(\phi)^2},\\&\frac{d}{d\phi}{}_ps(\phi)=2\frac{{}_pc(\phi)}{1+{}_pc(\phi)^2}\end{aligned}\qquad(4).$$

The structure of the previous identities marks the difference with the ordinary TF. The first of eqs. (4) can indeed be exploited to write the following differential equation

$$\frac{d}{d\phi}{}_pc(\phi)+\frac{1}{3}\frac{d}{d\phi}\left[{}_pc(\phi)\right]^3=-1\qquad(5)$$

Which, using the initial condition ${}_pc(0)=1$, can be reduced to the cubic equation

$$\left[{}_pc(\phi)\right]^3+3\,{}_pc(\phi)+3\phi-4=0\qquad(6)$$

We can accordingly conclude that the PTF are not transcendent but irrational functions, as easily inferred from the solution of eq. (6), which reads

[1] It is also easy to infer the derivatives of the inverse PTF functions, namely

$$\begin{aligned}&\frac{d}{dx}{}_pc^{-1}(x)=-(1+x^2),\\&\frac{d}{dx}{}_ps^{-1}(x)=\frac{2-x}{2\sqrt{1-x}}\end{aligned}$$

$$
\begin{aligned}
&{}_p c(\phi) = \sqrt[3]{R_+(\phi)} - \sqrt[3]{R_-(\phi)} \\
&R_\pm(\phi) = \sqrt{1 + \left(\frac{4-3\phi}{2}\right)^2} \pm \frac{4-3\phi}{2}
\end{aligned}
\qquad (7)
$$

Furthermore, on account of the fundamental identity (1), we find

$$
{}_p s(\phi) = 3 - \left[\sqrt[3]{R_+(\phi)^2} + \sqrt[3]{R_-(\phi)^2}\right] \qquad (8)
$$

it is worth noting that in correspondence of $\phi = \frac{4}{3}$ the parabolic cosine vanishes.

To understand the role of ϕ within the context of the parabolic trigonometry, we note that,in analogy with the circular case we can introduce the parabolic "pi" number as reported below

$$
\phi = \int_0^1 (1-\xi^2)\, d\xi = \frac{{}_p\pi}{2} \qquad (9a)
$$

We can therefore write

$$
\begin{aligned}
&{}_p c\left(\frac{{}_p\pi}{2}\right) = 0, \\
&{}_p s\left(\frac{{}_p\pi}{2}\right) = 1.
\end{aligned}
\qquad (9b)
$$

We have so far established that TPF are irrational functions and the corresponding "pi" is a rational number. Furthermore going back to the algebraic definition in terms of a cubic equation it can also be shown that the solution of the algebraic equation (6) can be obtained as [8-10]

$$
{}_p c(\phi) = -2\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{3\phi-4}{2}\right)\right) \qquad (10a)
$$

and, according to the fundamental PTF identity (1), we find fort the parabolic sine

$$
{}_p s(\phi) = 3 - 2\cosh\left(\frac{2}{3}\sinh^{-1}\left(\frac{3\phi-4}{2}\right)\right) \qquad (10b)
$$

which once combined yields the further identity

$$
{}_p c(\phi)\left({}_p s(\phi) - 4\right) = 3\phi - 4 \qquad (10c)
$$

We have so far recovered the essential aspects of the TPF properties derived in refs. [5,7], in the forthcoming section we will provide further elements of discussion.

Before closing this section we note that we have restricted our discussion to the first two quadrants of the coordinate system where $1-x^2 \geq 0$. The shape of the fundamental parabola is not suitable to define the PTF on all the four quadrants, we define therefore a closed curve whose superior external frontier is defined by the FP-I while the lower domain is provided by the FP-II , the final section is devoted to further comments in this direction.

2 Parabolic trigonometry and parabolic complex numbers

The parabolic and circular trigonometric functions (CTF) play, within certain extent, the same geometric role. It is therefore easy to derive the identities providing the respective link (see Fig. 2),

$$\begin{aligned} \sin(\varphi) &= \frac{{}_p s(\phi)}{{}_p I(\phi)}, \\ \cos(\varphi) &= \frac{{}_p c(\phi)}{{}_p I(\phi)}, \\ {}_p I(\phi) &= \sqrt{{}_p s(\phi)^2 + {}_p c(\phi)^2} \end{aligned} \tag{11}$$

where we have denoted by φ is the circular angle associate with the CTF specified by the coordinate of Q on the circumference of unitary radius . The Gudermann function [11]

$$\begin{aligned} \varphi &= gd(\phi) = \tan^{-1}({}_p t(\phi)), \\ {}_p t(\phi) &= \frac{{}_p s(\phi)}{{}_p c(\phi)} \end{aligned} \tag{12}$$

allows the transition from one to the other trigonometric frame.

The eqs. (11) can be inverted to find

$$\begin{aligned} {}_p s(\phi) &= \frac{\tan(\varphi)}{2\cdot\cos(\varphi)}\left[\sqrt{4-3\cdot\sin(\varphi)^2} - \sin(\varphi)\right] \\ {}_p c(\phi) &= \frac{1}{2\cdot\cos(\varphi)}\left[\sqrt{4-3\cdot\sin(\varphi)^2} - \sin(\varphi)\right] \end{aligned} \tag{13}$$

We can therefore obtain the values of parabolic sine and cosine in correspondence of the circular angles. We find, for example for $\varphi = \frac{\pi}{6}$ and of $\varphi = \frac{\pi}{4}$

$$\begin{aligned} &{}_p s(\phi_{\frac{\pi}{6}}) = \frac{1}{6}\left(\sqrt{13}-1\right), {}_p c(\phi_{\frac{\pi}{6}}) = \frac{1}{2\sqrt{3}}\left(\sqrt{13}-1\right), \\ &{}_p s(\phi_{\frac{\pi}{4}}) = {}_p c(\phi_{\frac{\pi}{4}}) = \frac{1}{2}\left(\sqrt{5}-1\right) \end{aligned} \tag{14}$$

The second of which is recognized as the inverse of the golden ratio.

The explicit form of the area sector in terms of ${}_p c(\phi)$ is obtained from eq. (5), which yields e.g.

$$\phi_{\frac{\pi}{6}} = \frac{\sqrt{3}}{54}(19 - 13\sqrt{13}) + \frac{4}{3},$$
$$\phi_{\frac{\pi}{4}} = \frac{5}{2}\left(1 - \frac{\sqrt{5}}{3}\right) \tag{15}$$

It is furthermore worth noting that from eq. (11) and from the PTF fundamental identity we find

$$\cos(\varphi)^2 X^2 + \sin(\varphi) X - 1 = 0,$$
$$X(\varphi) = {}_pI(\phi) \tag{16}$$

Which yields the following expression of ${}_pI$ as a function of φ

$$X(\varphi) = \frac{\sqrt{\sin(\varphi)^2 + 4\cos(\varphi)^2} - \sin(\varphi)}{2\cos(\varphi)^2} \tag{17}$$

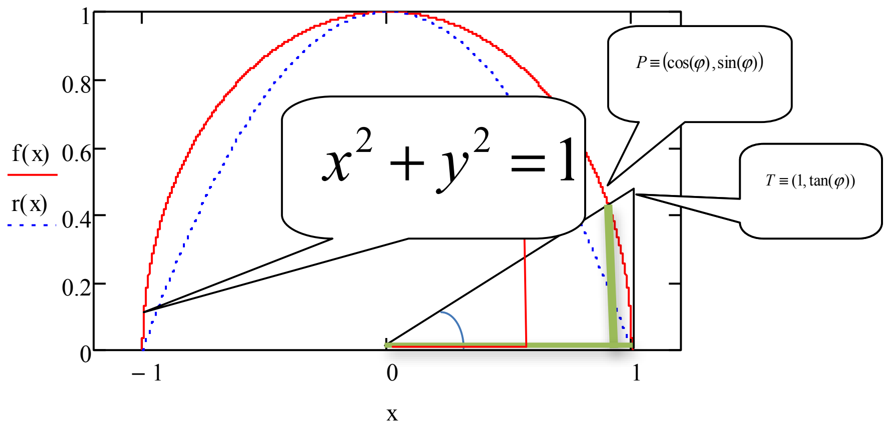

Fig. 2

PTF and TPF functions defined on parabolic and circular frames

The function ${}_pt(\phi)$ can be viewed as a kind of parabolic tangent, even though its role should be considered in more critical terms. Strictly speaking the circular tangent corresponds to the ordinate of the point T and is the segment tangent to the fundamental circumferences at the point of coordinates $(1,0)$. In the case of PTF the tangent to the FP at the point (1,0) is no more orthogonal to the x axis (see Fig. 3), strictly speaking the role of the tangent should be played by the ordinate of the point $R \equiv \left(\frac{2}{{}_pt(\phi)+2}, \frac{2\ {}_pt(\phi)}{{}_pt(\phi)+2}\right)$ while what we have denoted by ${}_pt(\phi)$ is the ordinate of the point $T \equiv (1,\ {}_pt(\phi))$.

In the case of ordinary trigonometry the line tangent to the fundamental curve in P is also perpendicular to that passing for the origin and P. The intersection of this line with the x-axis yields the secant, namely inverse of the cosine.

This is no more true for the parabolic case; a further view of the geometrical framework embedding the TPF is provided by the Fig. (4), where we point out that the intersection between the tangent to FP-I in P and the x-axis yields the point $S \equiv \left(\frac{1}{2} ({}_p sc((\phi) + {}_p c(\phi)), 0 \right)$ (with ${}_p sc(\phi) = \frac{1}{{}_p c(\phi)}$ being the parabolic secant).The lack of circular symmetry removes the degeneracy between tangent and perpendicular lines in P and implies the existence of a further point $L \equiv \left({}_p I(\phi) \; {}_p sc(\phi), 0 \right)$, the intersection between the x-axis and the line orthogonal to the parabola in P.

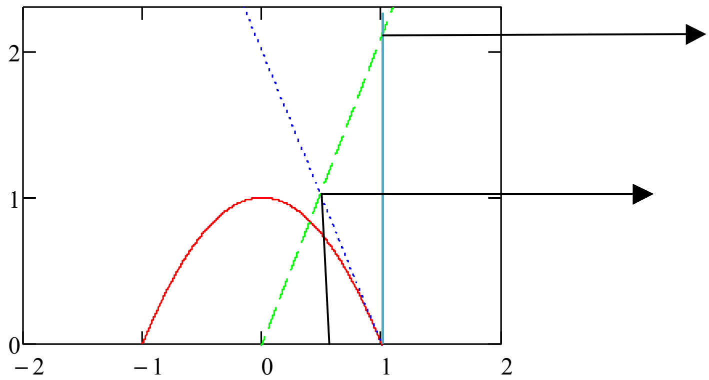

Fig. 3

Geometrical interpretation of parabolic tangent as the ordinate of the point $T \equiv \left(1, {}_p t(\phi)\right)$ ***and intersection point between the line tangent to FP-1 at the point (1,0)***

$$R \equiv \left(\frac{2}{{}_p t(\phi) + 2}, \frac{2 \, {}_p t(\phi)}{{}_p t(\phi) + 2} \right)$$

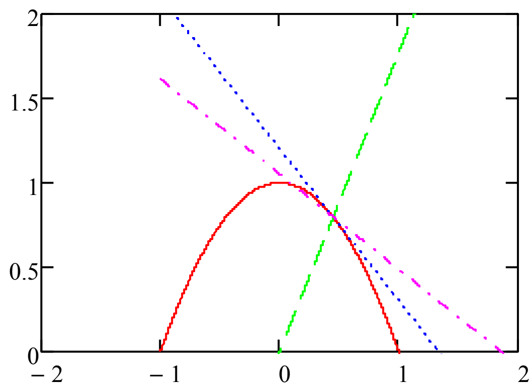

Fig. 4

Parabolic secants corresponding to the points S and Q on x-axis see text

Before closing this section we touch on the possibility of introducing polar parabolic form of a complex numbers. To this aim we define the complex quantity

$${}_pE(i\phi) = {}_p c(\phi) + i\sqrt{{}_p s(\phi)} \quad (18)$$

andnote that the relevant norm is

$$\left|{}_pE(i\phi)\right| = \sqrt{{}_pc(\phi)^2 + {}_ps(\phi)} = 1 \qquad (19)$$

It is furthermore evident that

$$\begin{aligned} &{}_pE(i\phi) = e^{i\Phi(\phi)}, \\ &\Phi(\phi) = \tan^{-1}\left(\frac{\sqrt{{}_ps(\phi)}}{{}_pc(\phi)}\right) \end{aligned} \qquad (20)$$

and the following Euler-like identities can be easily proved

$$\begin{aligned} &{}_pE(i\,{}_p\pi) = -1, \\ &{}_pE\left(i\frac{{}_p\pi}{2}\right) = i \end{aligned} \qquad (21)$$

Within this context we can introduce the parabolic rotation matrix

$$\hat{P}(\phi) = \begin{pmatrix} {}_pc(\phi) & -\sqrt{{}_ps(\phi)} \\ \sqrt{{}_ps(\phi)} & {}_pc(\phi) \end{pmatrix} \qquad (22)$$

preserving the Euclidean norm, but cannot be considered a rotation matrix in the strict sense since

$$\hat{P}(\phi_1) \circ \hat{P}(\phi_2) \neq \hat{P}(\phi_1 + \phi_2) \qquad (23)$$

further comments on the associated geometrical role will be presented elsewhere.

3 The para-hyperbolic functions

The TPF are In analogy to the PTF the Para-Hyperbolic-Functions (PHF) ${}_pch(\phi),\ {}_psh(\phi)$are defined as shown in Fig 5 as the coordinates of the point P on the FP of equation $\sqrt{x-1}$

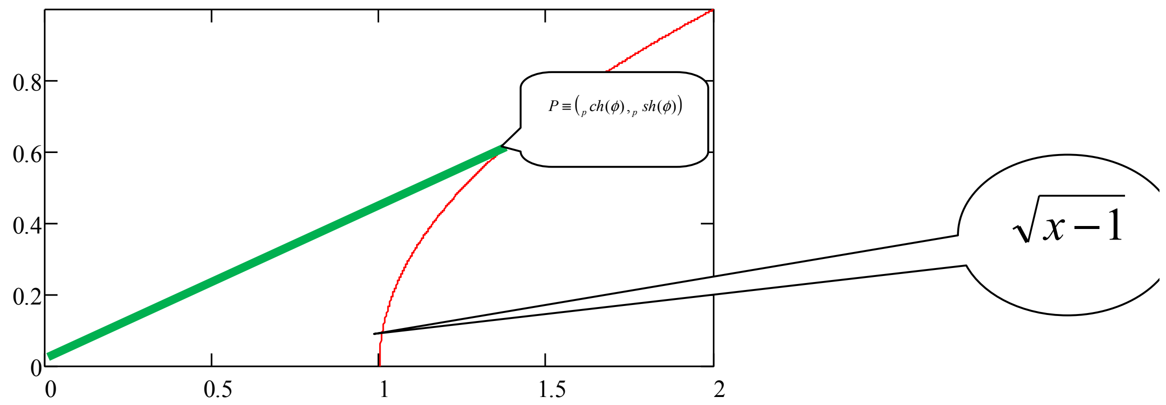

Fig. 5

Definition of the "Para-Hyperbolic-Functions" as the coordinates of the point $P \equiv \left({}_p ch(\phi), {}_p sh(\phi)\right)$***on the parabola of equation*** $y = \sqrt{1-x}$ with ϕ **being the area sector corresponding to twice the surface of the dashed region.**

The relevant properties are derived almost straightforwardly by following the same procedure as before, thus getting the fundamental identities

$$ {}_p ch(\phi) - {}_p sh(\phi)^2 = 1 \qquad (22)$$

and the associated derivatives

$$\begin{aligned} \frac{d}{d\phi}{}_p ch(\phi) &= \frac{2\;{}_p sh(\phi)}{7\;{}_p sh(\phi)^2 + 1}, \\ \frac{d}{d\phi}{}_p sh(\phi) &= \frac{1}{7\;{}_p sh(\phi)^2 + 1} \end{aligned} \qquad (23)$$

The second of the previous identities along with the condition ${}_p sh(0) = 0$, yields

$$7\;{}_p sh(\phi)^3 + 3\,{}_p sh(\phi) = 3\phi \qquad (24)$$

Which can be exploited to write the relevant explicit expression in the form

$$\begin{aligned} {}_p sh(\phi) &= 2\sqrt{\frac{1}{7}}\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{3}{2}\sqrt{7}\phi\right)\right), \\ {}_p ch(\phi) &= \frac{1}{7}\left[5 + 2\sinh\left(\frac{2}{3}\sinh^{-1}\left(\frac{3}{2}\sqrt{7}\phi\right)\right)\right] \end{aligned} \qquad (25)$$

In analogy with the discussion developed in the previous sections, we can introduce an associated family of para-hyperbolic numbers, such a discussion is however out of the scope of the present paper and is left for a dedicated investigation.

A very elementary application of the PTF is provided by their use for the description of the motion of a projectile with initial velocity v_0 ,along the longitudinal direction,under the action of the gravity. According to the previous discussion the parametric equations for the parabolic motion can be written as (see Fig. (7))

$$
\begin{aligned}
&X = v_0 T\, {}_p c\left(\frac{4}{3} - \phi\right),\\
&Y = y_0\, {}_p s\left(\frac{4}{3} - \phi\right),\\
&\phi = \frac{4}{3} - {}_p c^{-1}\left(\frac{t}{T}\right) = \qquad (26)\\
&= \left[\frac{4}{3} - \frac{1}{3}\left(\frac{5}{2} - \left(\frac{t}{T}\right)^2 \frac{t}{T}\right)\right]\left(1 - \frac{t}{T}\right)\\
&T = \sqrt{\frac{2 y_0}{g}}
\end{aligned}
$$

It is evident that the parameter ϕ is associated with the parabolic sector area, described by the projectile during its motion and T is the time taken to freely fall the height y_0.

Before closing the paper we mention a further point associated with the definition of PTF through a cubic equation, which can be viewed as an implicit function in terms of the parameter ϕ. The use of the Lagrange inversion formula [12] may provide a very effective tool, to get a series expansion in terms of ϕ.

We consider therefore eq. (24) and set

$$
\begin{aligned}
&\Sigma^3 - \Sigma + \sigma = 0,\\
&\Sigma = i\sqrt{\frac{7}{3}}\, {}_p sh(\phi), \sigma = i\sqrt{\frac{7}{3}}\phi
\end{aligned} \qquad (27)
$$

Where Σ can be viewed as a function of σ implicitly defined by the cubic identity (27), the Lagrange formula yields the series expansion of functions defined in implicit form and therefore, according to ref. [10], we can write

$$
\Sigma = -1 + \frac{\sigma}{2}\sum_{n=0}^{\infty}\frac{\Gamma(3n+1)\sigma^{2n}}{\Gamma(n+1)\Gamma(2n+2)} + \frac{\sigma^2}{2}\sum_{n=0}^{\infty}\frac{\Gamma(3n+5/2)\sigma^{2n}}{\Gamma(n+3/2)\Gamma(2n+3)} \qquad (28)
$$

Being ${}_p sh(\phi)$ a real function we can conclude from eqs. (28) and (27) that its series expansion reads

$$
{}_p sh(\phi) = \frac{\phi}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(3n)!}{n!(2n+1)!}\left(\sqrt{\frac{7}{3}}\phi\right)^{2n} \qquad (29).
$$

Which could be further elaborated in terms of hypergeometric series [10, 7].

In this paper we have shown that the concepts associated with the parabolic trigonometric functions hide interesting and worth to be studied aspects, which allow a more comprehensive understanding of their circular and hyperbolic relatives. An extensive discussion in applications will be discussed elsewhere.

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